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第12回偏微分方程式論
札幌シンポジウム

（代表者　久保田　幸　次）

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第12回偏微分方程式論
札幌シンポジウム

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代表者　久保田　幸　次

記

1. 日　時　　１９８７年７月２７日（月）　～　７月３０日（木）

2. 場　所　　北海道大学理学部数学教室　４－５０８室

3. 講　演

＊この時間は講演者を囲んでの自由な質問の時間とする予定です。

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Asymptotic Behaviors of Blow-Up Solutions for Semilinear Heat Equations

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次の初期値問題を考える.

(1) $$u_t=\Delta u+f(u)\ ,\quad t>0\ ,\quad x\in\Omega\ ,$$

(2) $$u(0,x)=u_0(x)\ ,\quad x\in\overline{\Omega}\ .$$

但し, Ω は $\mathbb{R}^N$ における滑らかな境界を持つ有界な領域（多次元の場合主に球領域 $\Omega=B_R=\{x;\ |x|<R\}$ $(0<R<\infty)$ を考える）, あるいは全空間 $\mathbb{R}^N$ とする.

Ω が有界領域の場合, その境界を $\partial\Omega=\Gamma$ として, 次の境界条件(BC1)または(BC2)付きの初期値境界値問題を考える.

(BC1) $$u(t,x)=0\quad \text{for}\quad x\in\Gamma\ .$$

(BC2) $$(\partial u/\partial n)+\beta u=0\quad \text{for}\quad x\in\Gamma\ .$$

ここでは, n は外向き法線ベクトル, $\beta=\beta(t)\geqq 0$ は連続関数で（ $\beta\equiv 0$ の場合 (BC2) は Neumann 境界条件となる）, 初期値 $u_0=u_0(x)\geqq 0$ は $C^1(\overline{\Omega})$ 関数である. また, 非線形項 f を $C^2(0,\infty)\cap C[0,\infty)$ 関数として, しかも

(3) $$f>0\ ,\ f'\geqq 0\ ,\ f''\geqq 0\ ,\quad \int_a^{\infty}(f(s))^{-1}ds<\infty\quad (\forall a>0)$$

が満たされるとする.

$u=u(t,x)$ を (1) の classical solution とし, T を

$$T\ =\ \sup\{\tau:\ u(t,x)\ \text{は}\ [0,\tau]\times\Omega\ \text{まで延長できる}\}$$

によって定義する.

こうして, $T<\infty$ の場合, $\limsup_{t\uparrow T}\|u(t,\cdot)\|_\infty=+\infty$ となり（但し, $\|\cdot\|_p$ $(1\leqq p\leqq\infty)$ は $L^p(\Omega)$ ノルムである）, u は爆発する(blow up)という. x_0 が u の爆発点 (blow-up point) とは, $n\to\infty$ のとき $t_n\uparrow T$, $x_n\to x_0$, $|u(t_n,x_n)|\to\infty$ となる点列 (t_n,x_n) があることである. なお, $S=\{x:\ x$ は u の爆発点$\}$ を u の爆発点集合, あるいは単に爆発集合 (blow-up set) という. 解の爆発条件, 或は爆発しない条件 (i.e. global existence) については, 既に多くの研究がなされてきた（例えば, 文献 [15],[16],[24],[25],[31],[32],[1],[29] 及びそれらの References など）. 初期値が（ある意味で）十分大きければ, 解 u(t,x) は必ず有限の時間内で爆発するということがよく知られている.

最近，解が爆発する場合で，その爆発集合及び漸近挙動についての研究が注目されている．

F.B.Weissler が 空間１次元で $\Omega=(-R,R)$ ， $f(u)=u^p$ （$p>2$，十分大）および Dirichlet 条件のもとで，ある特別な初期値に対して，$S=\{0\}$ ，つまり一点のみで爆発することを示した（[33]）．その後，A.FriedmanとB.McLeod（[13]）が Dirichlet 境界条件のもとで，ごく一般的な有界領域Ωに対して，SがΩの内部にある Compact subset であることを示した．[13] では，球領域Ωで Dirichlet 条件を満たして中心のみで最大値を取る回転対称かつ半径方向単調減少の解に対して，爆発集合は中心一点からなることも証明された．また，一次元 $N=1$ の場合については，初期値が二つの最大値点を持つ（i.e. 2-peakの）場合，Sは一点か，二点か，または一つの閉区間からなることも示された．H.Fujita と Y.-G.Chen は，初期値の対称性を仮定し，それぞれDirichlet条件，Neumann条件あるいは第三種の境界条件（Robin条件）のもとで，二つの最大値点を持つ初期値から出発した解が高々二点で爆発することを証明した([17])．また，Y.-G.Chen は，多次元空間の球領域で回転対称の解に対して，上述の三種類の境界条件それぞれ（あるいはΩが全空間）のもとで，(1) 初期値が半径方向単調減少ならばSは一点からなる；(2) 初期値 $u_0(x)=\phi(|x|)=\phi(r)$ が半径 r の関数として区間 $(0,R)$ に最大値点を一つのみ持つとすれば，Sは一点またはある球面 S^{N-1}から構成される，ということを示した([8])．最近，L.A.CaffarelliとA.Friedman が，Dirichlet 条件のもとで（対称性を仮定せずに）初期値が二つの最大値点を持つとすればSは一点か二点からなることを証明した([4])．

一方，$f(u)=|u|^{p-1}u$ $(p>1)$ の場合，爆発解 u の爆発点の近傍での漸近挙動については，Y.Giga とK.V.Kohn が自己相似（self-similar）変換を使って調べた（[18]，[21]～[23]）．それによると，$k=(p-1)^{-1/(p-1)}$ とおき，$p\leqq(N+2)/(N-2)$ または $n\leqq 2$ として，$|u(t,x)|\cdot(T-t)^{1/(p-1)}$が有界ならば，任意の $a\in\Omega$ に対して

$$\lim_{t\uparrow T}u(t,x\sqrt{T-t}+a)\cdot(T-t)^{1/(p-1)}=\pm k \text{ または } 0$$

となる．さらに，Giga-Kohn の最近の結果（[23]）によると，同じ条件のもとで以上の極限が 0 ならば $a\in\Omega$ は爆発点ではないことが分かった．なお，Friedman と McLeod ([13])は Dirichlet境界条件(及び第三種の境界条件)に対して，Chen([8])は Neumann 境界条件に対して，$|u(t,x)|\cdot(T-t)^{1/(p-1)}$ が有界であることを示した．

解の爆発する場合の漸近挙動について，Chen は数値実験及び数値解析の方法で差分解の爆発の様子を調べた([7],[9])．爆発問題に適用する差分法及び有限要素法については [27],[28],[7],[9] で研究された．

ここでは，初期関数 u_0 が爆発条件を満たして解が爆発する場合，その漸近挙動及び爆発集合について最近得られた結果を報告する．

非線形項 $f(u)$ については，次の条件 (4) が満たされるとする．

(4)　f に対して，次の ①～④ を満たす関数 F が存在する．

① $s>0$ ならば，$F(s)>0$, $F'(s)>0$, $F''(s)\geqq 0$;

② 任意の $M>0$ に対して，

$$f'(u)F(u)-f(u)F'(u)\geqq MF(u) \quad \text{for all} \quad u\geqq C(M)$$

となるような定数 $C=C(M)>0$ が存在する；

③ 任意の十分大の $v>0$ に対して，

$$f'(u)F(u)-f(u)F'(u)\geqq cF(u)F'(u) \quad \text{for all} \quad u\geqq v$$

となるような定数 $c=c(v)>0$ が存在する；

④ 任意の $a>0$ に対して，$\int_a^{\infty}(F(s))^{-1}ds<+\infty$.

注意：条件 (4) は，例えば，$f(u)=(u+\mu)^P$ $(P>1,\ \mu\geqq 0)$, $f(u)=\exp(\lambda u)$ $(\lambda>0)$, $f(u)=u^p+u^q$ $(p>1$, $q>1)$ などのような関数によって満たされて，適用できる．(4) が満たされるような十分条件としては，“$\exists q\in(0,1)$ such that $F(u)=(f(u))^q$ は (3) を満たす” という条件が挙げられる([17])．

われわれの主要結果は，次の定理である．([17],[8])

<u>定理</u>　以上の仮定のもとで，次の（Ⅰ）～（Ⅷ）が成立する．

（Ⅰ） $N\geqq 1$, $\Omega=B_R$, $u_0(x)=\phi(|x|)\geqq 0$, (BC2) を仮定する（$\beta\equiv 0$, つまり Neumann境界条件でもよい）．$\beta\equiv 0$ しかも $\phi\equiv$constant in $[0,R]$ ならば，$S=\overline{\Omega}$; $\phi'\leqq 0$, $\phi'\not\equiv 0$ in $(0,R)$ ならば，$S=\{0\}$.

（Ⅱ） $N=1$, $\Omega=(-R,R)$, $u_0(x)=u_0(-x)$, しかも $u_0(x)$は $(-R,R)$ において一つまたは二つの最大値点(1-peak or 2-peak)を持つとする．

① (BC1) を仮定すれば，$S=\{0\}$, または $S=\{-a,a\}$, $0<a<R$;

② (BC2) を仮定し，しかも $u_0(x)\geqq u_0(R-x)$ for $x\in(0,R/2)$ ならば，$S=\{|x|=a\}$, $0\leqq a\leqq R/2$ (i.e. Sは１点または２点からなる)；

③ (BC2) で $\beta\equiv 0$, しかも $u_0(x)\leqq u_0(R-x)$ for $x\in(0,R/2)$ ならば，$S=\{-a,a\}$, $R/2\leqq a\leqq R$ (i.e. S は２点から構成される)．

（Ⅲ） (BC1), $N\geqq 2$, $\Omega=B_R$, $u_0(x)=\phi(|x|)\geqq 0$. ある $\gamma\in(0,R)$に対して，

(5)　$\phi'(r)\geqq 0$ for $0<r<\gamma$, $\phi'(r)\leqq 0$ for $\gamma<r<R$,

しかも $\phi'\not\equiv 0$ for $r\in(0,\gamma)$ or $r\in(\gamma,R)$

とすれば，Sは１点($S=\{0\}$)または N-1次元の球面 (i.e. $S=\{r=a\}$, $0<a<R$) から構成される．

（Ⅳ） (BC2), $N\geqq 2$, $\Omega=B_R$, $u_0(x)=\phi(|x|)\geqq 0$ が (5) を満たすとして，さらに $\phi(r)\geqq\phi(R-r)$ in $(0,R/2)$ と仮定すれば，Sは１点 ($S=\{0\}$)または N-1

次元の球面（$S=\{r=a\}$，$0<a<R/2$）から構成される．

（V）（IV）において，$\beta\equiv 0$（i.e. Neumann条件の場合），$\phi(r)\equiv\phi(R-r)$ for r in $(0,R/2)$ とする．

① $N=1$ ならば，$S=\{-R/2,R/2\}$；

② $N\geqq 2$ ならば，$S=\{r=a\}$，但し，$0\leqq a<R/2$（i.e. S は $B_{R/2}$ の中に含まれる）．

（VI） $N\geqq 1$，Ω を有界な領域，境界条件（BC2）において $\beta\equiv 0$，非線形項を $f(u)=|u|^{p-1}u$ $(p>1)$ として，さらに $u_0\in C^2$，$\Delta u_0+f(u_0)\geqq 0$ とすれば，次の二つの式が成り立つ。

(6) $\forall\varepsilon>0$，$\exists C>0$ such that

$$u(t,x)\leqq C(T-t)^{-1/(p-1)},\quad (t,x)\in[\varepsilon,T)\times\Omega,$$

(7) $\exists c>0$ such that $\|u(t,\cdot)\|_{\infty}\geqq c(T-t)^{-1/(p-1)}$，$t\in(0,T)$．

（VII） （VI）の仮定のもとで，$p\leqq(N+2)/(N-2)$ または $n\leqq 2$ として，ある $a\in\Omega$ に対して，$\lim_{t\uparrow T}u(t,a)=+\infty$ とすれば，$x\in\Omega$ に対して $\lim_{t\uparrow T}u(t,x)=+\infty$ となるための必要十分条件は，$\lim_{t\uparrow T}\{u(t,x)/u(t,a)\}=1$ である．

（VIII） $N\geqq 1$，$\Omega=\mathbb{R}^N$，$u_0(x)=\phi(|x|)\geqq 0$ とする．

① $\phi(r)\equiv$constant in $(0,\infty)$ ならば，$S=\mathbb{R}^N$；

② $\phi'(r)\leqq 0$ しかも $\phi'\not\equiv 0$ in $(0,\infty)$ ならば，$S=\{0\}$． ∎

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SUPPORT OF THE SOLUTION OF A QUASILINEAR DIFFUSION EQUATION IN R^N

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1. Statement of the phenomenon. Let us consider the Cauchy problem for the quasilinear diffusion equation :

(1) $$u_t = \Delta\phi(u) , \ t > 0 , \ x \in R^N ;$$

(2) $$u(0,x) = u_0(x) , \ x \in R^N .$$

In the case of $\phi(r) = r$, the solution of the linear problem (1) - (2) is given explicitly by $e^{t\Delta}u_0(x)$:

$$(4\pi t)^{-N/2}\int_{R^N} e^{-|x - y|^2/(4t)} u_0(y)\, dy$$

and spreads out in R^N for any $t > 0$ even if

(3) $$u_0 \in L^\infty(R^N) \ \text{ and } \ u_0(x) = 0 \ \text{ for } \ |x| \geqq R , \ R > 0 .$$

However the situation is quite different if $\phi(r) = r^3$, or more generally, if the function ϕ satisfies

(4) ϕ is a nondecreasing, locally Lipschitz continuous function on R^1 with $\phi(0) = 0$;

(5) the range of ϕ coincides with R^1 and the integral $\int_0^r 1/\phi^{-1}(s)\, ds$ exists for each $r \in R^1$.

The purpose of this talk is to show that in this case the solution $u(t,x)$ of (1) - (2) with (3) stays compactly supported forever :

$$u(t,x) = 0 \quad \text{a.e.} \quad |x| \geqq R + C\, t^{1/2} \quad \text{for} \quad t \geqq 0$$

with a constant $C > 0$ depending on ϕ and $\|u_0\|_\infty$. This implies that the problem (1) - (2) describes a "finite speed propagation" process and loses its parabolicity.

This kind of phenomenon has already been found out in the case of $N = 1$. Indeed the lateral boundary, called the interface, of the support has been shown to consist of two monotone Lipschitz continuous curves satisfying a certain differential equation. Even the regularity and the asymptotic bahavior of it have been studied extensively (see e.g. [5,6]).

However the study in the case of $N > 1$ has been somehow neglected. To the knowledge of the present speaker, little is known about support of the solution of the Cauchy problem (1) - (2) under (3). The main difficulty in this talk is that no exact solution of (1) - (2) can be found, instead only a "good function" $v(t,x)$ for which

$$v_t - \Delta\phi(v) \quad \text{is nonnegative and "small"} .$$

It is convenient to denote the solution of (1) - (2) by $u(t,x) = S_\phi(t)u_0(x)$, where $\{S_\phi(t) : t > 0\}$ is the associated semigroup generated in $L^1(R^N)$ by $\Delta\phi$ in the sense of Crandall and Liggett [3]. This semigroup was constructed first by Bénilan, Brézis and Crandall [1] under a condition weaker than (4) , and recently by the speaker [8] under (4) by a simpler method.

THEOREM. Let u_0 satisfy (3) , and ϕ satisfy (4) and (5) . Then for all $t \geqq 0$

$$S_\phi(t)u_0(x) = 0 \quad \text{a.e.} \quad |x| \geqq R + 2\, C_0\, t^{1/2}$$

with $$C_0 = \left\{\int_0^{\phi(\|u_0\|_\infty)} 1/\phi^{-1}(s)\, ds\right\}^{1/2} .$$

REMARKS. The theorem coincides with a result for $N = 1$ of Knerr [5, Theorem 8.4] and similar to a result of Diaz Diaz [4, Theorem I] for an initial-boundary value problem in a bounded subdomain of R^N . See also Schatzman [7] for a result concerning (1) - (2) with a sort of absorption.

2. A "good function" with compact support.

We shall first look for a good function with compact support in R^N satisfying, instead of (1) ,

(6) $$v_t = \Delta\phi(v) + g(t,x) \ , \ t \geqq 0$$

with a nonnegative valued and "small" function $g(t,x)$.

Let us consider, under (4) and (5) , the function $v = v(t,x)$ determined by

$$\int_0^{\phi(v)} 1/\phi^{-1}(s)\, ds = \max(0,\, a - |x|^2/4(t+b)) ,$$

where a and b are positive constants. Set

$$y(r) = \phi^{-1}(z^{-1}(r)) \quad \text{with} \quad z(r) = \int_0^r 1/\phi^{-1}(s)\, ds .$$

Then y is a nondecreasing continuous function on $[0,\infty)$ with

(7) $$(d/dr)\phi(y(r)) = y(r) \quad \text{and} \quad y(0) = 0 \; ,$$

and our function v can be written as

$$v(t,x) = y((a - |x|^2/4(t+b))^+) \; ,$$

where $r^+ = \max(0,r)$ for $r \in R^1$. Evidently v is nonnegative valued and compactly supported in R^N :

(8) $$v(t,x) = 0 \quad \text{for} \quad |x|^2 \geqq 4a(t+b) \quad \text{with} \quad t \geqq 0 \; .$$

LEMMA 1. For each fixed $T > 0$,

(i) v belongs to $C([0,T];L^1(R^N)) \cap L^\infty((0,T) \times R^N)$;

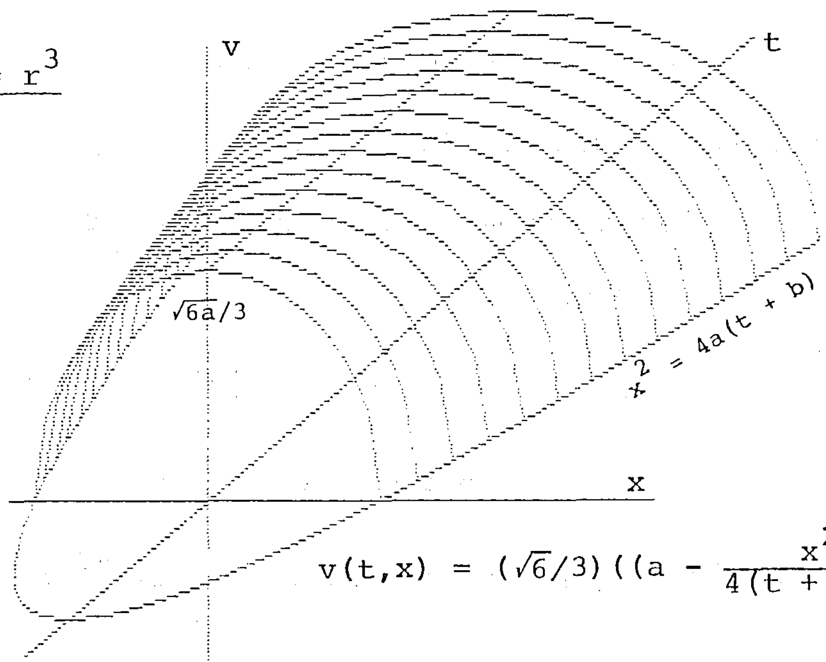
(ii) v is a solution of (6) in $D'((0,T) \times R^N)$ with

$$g(t,x) = y((a - |x|^2/4(t+b))^+)\cdot N/2(t+b) \; ;$$

(iii) g is nonnegative valued and belongs to

$$L^1(0,T;\, L^1(R^N) \cap L^\infty(R^N)) \; .$$

$\phi(r) = r^3$



$$v(t,x) = (\sqrt{6}/3)((a - \frac{x^2}{4(t+b)})^+)^{1/2}$$

PROOF. We can easily verify (i) by Lebesgue's theorem. An elementary calculus by (7) shows

$$\sum_{i=1}^{N} (\partial/\partial x_i) \int_0^t (\partial/\partial x_i) \phi(v) \; dr = [v(r,x)]_0^t - \int_0^t g(r,x) \; dr$$

for $t \geqq 0$ and $x \in R^N$. So multiplication by ψ_t with $\psi \in C_0^\infty((0,T) \times R^N)$ and integration over $(0,T) \times R^N$ gives

$$\int_0^T \int_{R^N} (v\psi_t + \phi(v)\Delta\psi + g(t,x)\psi) \; dxdt = 0 .$$

The proof of (iii) will be clear. Q.E.D.

Combining Lemma 1 with general or particular properties of the semigroup $\{S_\phi(t) : t > 0\}$ and abstract solutions in $L^1(R^N)$ of (6), we can deduce the comparison result :

LEMMA 2. If

$$0 \leqq u_0(x) \leqq v(0,x) \quad \text{for a.a. } x \in R^N ,$$

then

$$0 \leqq S_\phi(t)u_0(x) \leqq v(t,x) \quad \text{for a.a. } x \in R^N$$

and all $t \geqq 0$.

We shall put off its proof to the next section, and give here the proof of Theorem.

PROOF OF THEOREM. Assume in addition that $u_0(x)$ is nonnegative for all $x \in R^N$ without loss of generality.

Let a and b satisfy $a > R^2/4b$ and

(9) $$\|u_0\|_\infty = y(a - R^2/4b) .$$

Then $v(0,x) = y((a - |x|^2/4b)^+)$ is not smaller than $u_0(x)$

for a.a. $|x| < R$ and obviously for $|x| \geqq R$, that is,

$$0 \leqq u_0(x) \leqq v(0,x) \quad \text{for a.a. } x \in R^N .$$

So we have by Lemma 2

$$0 \leqq S_\phi(t)u_0(x) \leqq v(t,x) \quad \text{for a.a. } x \in R^N$$

and all $t \geqq 0$. Therefore we see from (8) and (9) that $S_\phi(t)u_0(x)$ vanishes for all $t \geqq 0$ and a.a. $x \in R^N$ such that

$$|x|^2 \geqq 4(t + b)\{R^2/4b + z(\phi(\|u_0\|_\infty))\} , \; b > 0 .$$

We have thus only to find out the envelope. Choosing $b = (R/2)\{t/z(\phi(\|u_0\|_\infty))\}^{1/2}$ for $t > 0$, we can conclude

$$S_\phi(t)u_0(x) = 0 \quad \text{a.e. } |x| \geqq R + 2\{z(\phi(\|u_0\|_\infty))t\}^{1/2}$$

for all $t \geqq 0$. Q.E.D.

3. Comparison of solutions of two kinds. We shall finish this talk by a short explanation of Lemma 2. Observe, under (4), the approximation of $\Delta\phi$: For each integer $m > 0$

$$A_\phi^h = k^{-1}(e^{k\Delta} - I)\phi(\cdot) \quad \text{as } h \downarrow 0$$

where $k = h \cdot \sup_{|r|,|s| \leqq m} (\phi(r) - \phi(s))/(r - s)$.

We found [8] that this is nice enough to guarantee the existence for $\lambda > 0$ of $(I - \lambda\Delta\phi)^{-1}$ and the generation of $\{S_\phi(t): t > 0\}$ by a simple method. This is again convenient to deal with abstract solutions in $(L^1(R^N), \|\cdot\|_1)$ of

$(1)_f$ $$u_t = \Delta\phi(u) + f(t,x) , \; 0 < t \leqq T$$

with (2) under the condition :

$$(3)_f \qquad u_0 \in L^1(R^N) \cap L^\infty(R^N) \quad \text{and} \quad f \in L^1(0,T;L^1(R^N) \cap L^\infty(R^N)) .$$

The "integral solution" u of $(1)_f - (2)$ is indeed approximated uniformly on $[0,T]$ by the "strong solution" u_h of

$$(1)_f^h \qquad u_t = A_\phi^h u + f(t,x) , \; 0 < t \leqq T$$

with (2) , which is equivalent to the equation in $L^1(R^N)$

$$u(t) = e^{-t/h}u_0 + h^{-1}\int_0^t e^{-(t-r)/h} C_h u(r)\, dr$$
$$+ \int_0^t e^{-(t-r)/h} f(r)\, dr \text{ (cf.[8, p.513])}$$

where $C_h = I + h A_\phi^h$. This admits a unique solution u_h in $C([0,T];\, L^1(R^N) \cap L^\infty(R^N))$ satisfying, for $0 \leqq t \leqq T$,

$$(10) \qquad \|u_h(t)\|_\infty \leqq \|u_0\|_\infty + \int_0^t \|f(r)\|_\infty\, dr \; (\leqq m)$$

by a well-known fixed point theorem. Clearly u_h satisfies

$$(11) \qquad \int_0^T \int_{R^N} (u_h\psi_t + \phi(u_h)k^{-1}(e^{k\Delta} - I)\psi + f(t,x)\psi)\, dxdt = 0$$
$$\text{for all } \psi \in C_0^\infty((0,T) \times R^N) .$$

<u>LEMMA 3.</u> Under $(3)_f$ the following are equivalent :

(i) u is the integral solution of $(1)_f - (2)$;

(ii) u belongs to $C([0,T];L^1(R^N)) \cap L^\infty((0,T) \times R^N)$ and satisfies

$$\begin{cases} u(0,x) = u_0(x) , \\ \displaystyle\int_0^T \int_{R^N} (u\psi_t + \phi(u)\Delta\psi + f(t,x)\psi)\, dxdt = 0 \\ \qquad \text{for all } \psi \in C_0^\infty((0,T) \times R^N) . \end{cases}$$

<u>PROOF.</u> The implication (i) $\to$ (ii) will be clear

from (10) and (11), and (ii) $\to$ (i) follows from the uniqueness theorem given in [2, Theorem 1]. Q.E.D.

<u>LEMMA 4.</u> Let u_0, $\hat{u}_0$; f, $\hat{f}$ satisfy $(3)_f$. Then the corresponding integral solutions u, $\hat{u}$ of $(1)_f$ – (2) satisfy

$$\|(u(t) - \hat{u}(t))^+\|_1$$
$$\leq \|(u_0 - \hat{u}_0)^+\|_1 + \int_0^t \|(f(r) - \hat{f}(r))^+\|_1 \, dr .$$

<u>PROOF.</u> The lemma is true for the corresponding strong solutions u_h, $\hat{u}_h$ of $(1)_f^h$ – (2) since

$$\int_{R^N} \operatorname{sgn}^+(u - v) \cdot (A_\phi^h u - A_\phi^h v) \, dx \leq 0 ,$$

where $\operatorname{sgn}^+ r = 1$ $(r > 0)$ and 0 $(r \leq 0)$. Q.E.D.

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Navier-Stokes Flow Down an Inclined Plane

Yoshiaki TERAMOTO

§1 Introduction

The motion of viscous incompressible flow down an inclined plane under gravity is governed by the following equation

$$
(1.1)\qquad \begin{aligned} &\partial_t \bar{v} - \nu\Delta\bar{v} + (\bar{v},\nabla)\bar{v} + \nabla\bar{p} = \bar{f} \ , \\ &\operatorname{div} \bar{v} = 0 \qquad\qquad \text{in } \Omega(t) \ , \end{aligned}
$$

where $\Omega(t) = \{ (y_1,y_2,y_3) \ ; \ -b < y_3 < \eta(y_1,y_2,t) \}$ is the domain occupied by the fluid at time t. $\bar{f}=(g\sin\alpha,0,-g\cos\alpha)$ is the external force, where g is the gravity constant and α $(0<\alpha<\pi/2)$ is the angle of the inclination of the bottom plane $S_B = \{ y_3 = -b \}$. We set $\bar{p} = p + p_0 - y_3 g\cos\alpha$. p_0 is the atomospheric pressure assumed to be constant. ν is the coefficient of viscosity. The density is set to be one. At the free surface $y_3 = \eta(y_1,y_2,t)$ $((y_1,y_2)\in\mathbb{R}^2, \ t>0)$ $\bar{v}$, p and η satisfy

$$
(1.2)\qquad \partial_t\eta = \bar{v}_3 - \bar{v}_1\partial\eta/\partial y_1 - \bar{v}_2\partial\eta/\partial y_2 \ ,
$$

$$
(1.3)\qquad \begin{aligned} &(p - g\cos\alpha)n_i - \nu(\partial v_i/\partial y_j + \partial v_j/\partial y_i)n_j \\ &\qquad + \beta\nabla_F\{(1+|\nabla_F\eta|^2)^{-1/2}\nabla_F\eta\}n_i = 0 \ , \quad i=1,2,3, \end{aligned}
$$

where $\nabla_F = (\partial/\partial y_1,\partial/\partial y_2)$, and $n = (n_1,n_2,n_3)$ denotes the outward unit normal of the free surface at $y_3=\eta(y_1,y_2,t)$. β is the coefficient of the surface tension. On the bottom S_B the fluid satisfies the adherence condition

$$(1.4) \qquad \bar{v} = 0 \qquad \text{on } S_B .$$

We are also given the initial condition at $t = 0$

$$(1.5) \qquad \begin{aligned} \eta &= \eta_0(y_1, y_2) , \qquad (y_1, y_2) \in R^2 \\ \bar{v} &= v_0(y) , \qquad y \in \Omega(0) , \end{aligned}$$

where $\Omega(0) = \{(y_1, y_2, y_3) \; ; \; -b < y_3 < \eta(y_1, y_2) \}$.

If we assume that $\eta \equiv 0$ and that the velocity is parallel to the inclined plane and depends only on the depth, we can easily obtain the steady flow

$$(1.6) \qquad w = (w_1, w_2, w_3) = ((2\nu)^{-1} g\,sin\alpha (b^2 - x_3{}^2), 0, 0) .$$

The associated pressure is $p_0 - x_3 g\,cos\alpha$. We denote the fluid region of this unperturbed flow by $\Omega \equiv \{(x_1, x_2, x_3); \; -b < x_3 < 0 \}$. In the sequel we reduce (1.1)-(1.5) to the problem on the fixed domain Ω, and are concerned with global in time solutions to the equations governing the disturbances from the unperturbed flow. The approach used here is the one employed by Beale in [1].

§2 Reduction to the fixed domain

Let $\partial\Omega \equiv S_F \cup S_B$, where $S_F = \{ x_3 = 0 \}$ and $S_B = \{ x_3 = -b \}$.

We regard η as the function defined on $S_F \times \{t \geq 0\}$, then extend this for $x_3 < 0$ as follows :

$$(2.1) \qquad \bar{\eta}(x_1, x_2, x_3, t) = \mathcal{F}^{-1}(\exp(|\xi| x_3)\eta^{\sim}(\xi, t)),$$

where $\eta^{\sim}(\xi, t)$ is the Fourier transform with respect to the horizontal variables (x_1, x_2) and $\mathcal{F}^{-1}$ is its inverse. For each $t > 0$ we define the transformation θ on Ω onto $\Omega(t) = \{ y ; -b < y_3 < \eta(y_1, y_2, t) \}$ by

$$(2.2) \qquad y = \theta(x, t) = (x_1 , x_2 , \bar{\eta} + x_3(1 + \bar{\eta}/b)) .$$

The velocity field $\bar{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3)$ and the scalar pressure p on $\Omega(t)$ are defined from the vector field v and the scalar q on Ω by

$$(2.3) \qquad \begin{aligned} &\bar{v}_i(\theta(x,t),t) = \theta_{i,x_j} v_j / J \equiv \alpha_{ij} v_j , \quad i=1,2,3 , \\ &p(\theta(x,t),t) = q(x,t). \end{aligned}$$

Here (θ_{i,x_j}) is the Jacobian matrix and $J = \det(\theta_{i,x_j})$. Note that $\nabla_y \cdot \bar{v} = 0$ in $\Omega(t)$ if and only if $\nabla \cdot v = 0$ in Ω . We set $v = u + w$. Using the fact that $(\theta_{i,x_j}) = (\delta_{ij})$ when $\eta \equiv 0$, we can rewrite (1.1)-(1.4) to the problem for (η, u, q) on the unperturbed domain Ω as follows :

$$(2.4) \qquad \partial_t \eta = u_3 \qquad \text{on } S_F ,$$

$$(2.5) \qquad \partial_t u - \nu\Delta u + (w,\nabla)u + (u,\nabla)w + \nabla q = F_0(\eta,u,q) \quad \text{in } \Omega ,$$

$$(2.6) \qquad \operatorname{div} u = 0 \qquad \text{in } \Omega ,$$

$$(2.7) \qquad u = 0 \qquad \text{on } S_B ,$$

$$(2.8) \qquad \partial_j u_3 + \partial_3 u_j = F_j(\eta, u) \quad \text{on } S_F , \quad j = 1,2 ,$$

$$(2.9) \qquad q - 2\nu\partial_3 u_3 - (g\cos\alpha - \beta\Delta_F)\eta = F_3(\eta, u) \quad \text{on } S_F .$$

Here F_j ($j=0,..3$) on the right hand sides contain the quadratic and higher terms of η, u, q and their derivatives.

§3 The result

To state the result we introduce some notations : $H^r(\Omega)$ is the usual Sobolev space. $K^r(\Omega \times R^+) = H^0(R^+;H^r(\Omega)) \cap H^{r/2}(R^+;H^0(\Omega))$. We say that η belongs to $\tilde{K}^r(R^2 \times R^+)$ if the following conditions hold :

$\eta \in K^r(R^2 \times (0,T))$ for any $T > 0$. $\eta = \eta_1 + \eta_2$ such that η_1 $K^r(R^2 \times R^+)$ and η_2 is the Fourier transform in space-time variables of L^1 functions of bounded support. (See [1] for details.)

We consider the intial value problem for (2.4)-(2.9) with the initial data

$$(3.1) \qquad \begin{aligned} &\eta = \eta_0(x_1, x_2) \qquad (x_1, x_2) \in R^2 , \\ &u = u_0(x) \qquad x \in \Omega . \end{aligned}$$

Suppose the compatibility conditions on the initial data :

$$\operatorname{div} u_0 = 0 \quad \text{in } \Omega , \quad u_0 = 0 \quad \text{on } S_B ,$$
$$\partial_j u_{0,3} + \partial_3 u_{0,j} = \gamma_j \quad \text{on } S_F \quad (j=1,2) ,$$

where γ_j (j=1,2) are the quantities written in terms of η_0 .
Further, assume that the Reynolds number of the unperturbed flow

$$2^{-1}\nu^{-2}b^3 g \sin\alpha$$

is sufficiently small. The result is the following

THEOREM. Suppose the conditions stated above hold. Let $3 < r < 7/2$. There exists $\delta > 0$ such that if η_0 and u_0 satisfy

$$|\eta_0|_{H^r(R^2)} + |u_0|_{r-1/2} < \delta ,$$

then there exists a global solution (η, u, q) of (2.4)-(2.9), which satisfies

$$\eta \in \tilde{K}^{r+1/2}(R^2 \times R^+) , \ u \in K^r(\Omega \times R^+) ,$$
$$\nabla q \in K^{r-1/2}(\Omega \times R^+).$$

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Microlocal orders of singularities for distributions and applications

Sadao Miyatake

§1. Orders of singularities in R^n.

Let f be a distribution with compact support. Then as is well-known, there exists an integer $m = m(f)$ such that f belongs to H^{-2m}. More precisely f is represented by $f = (1 - \triangle)^m f_o$, where f_o belongs to $L^1 \cap L^2$. This implies

$$(1 + |\xi|^2)^{-m} \hat{f}(\xi) \in \mathcal{B}^0 \cap L^2, \quad \text{for } f \in \mathcal{E}'(R^n) .$$

Therefore we can use

Definition 1.1. For $f \in \mathcal{E}'$, the order of singularities are considered in the following two ways.

$$OS_1(f) = \inf \left\{ r \in R, \ \lim (1+ |\xi|)^{-r} \hat{f}(\xi) = 0 \right\} ,$$

$$OS_2(f) = \inf \left\{ r \in R, \ f \in H^{-r} \right\} .$$

First we remark the following lemma:

Lemma 1.1. For any $f \in \mathcal{E}'$ and any $\alpha \in C_o^\infty$, we have

$$(1.1) \qquad OS_k(\alpha f) \leq OS_k(f) \quad , \quad k = 1,2.$$

Now we introduce the following definition:

Def.1.2. $\{\alpha_n\}$ is said to be a localizing sequence to $x_o \in R^n$ if $\alpha_n \in C_o^\infty$ satisfy $\alpha_n(x_o) \neq 0$ and $\operatorname{supp} \alpha_n \longrightarrow x_o$ in the sense that for given $\varepsilon > 0$ there exists N such that $\operatorname{supp} \alpha_n \subset \{x; |x-x_o| < \varepsilon\}$ for all $n > N$.

Here we state

Theorem 1.1. Let f belong to $\mathcal{D}'$. For any localizing sequence $\{\alpha_n\}$ to x_o it holds

$$(1.2) \qquad \lim_{n \to \infty} OS_k(\alpha_n f) = \inf_{\alpha \in C_o^\infty, \ \alpha(x_o) \neq 0} OS_k(\alpha f) \quad , \quad k=1,2.$$

Now we describe

Definition 1.3. For $f \in \mathcal{D}'$, the orders of singularities at x_0, which we denote by $OS_k(f;x_0)$, $k=1,2$, are defined by the numbers given in (1.2).

Example. $OS_1(\delta;0) = 0$, $OS_2(\delta;0) = n/2$ for δ in R^n and $OS_1(\text{p.v.}\ \frac{1}{x}$ in R^1 ; $0) = 0$. Later we use $OS_1(t_+^m;0) = -(m+1)$, where $t_+^m = t^m, t > 0$ and $=0, t \leq 0$.

Remark 1.1. For $f \in \mathcal{E}'$, $OS_k(f;x)$, $k=1,2$, are upper semi-continuous in x and we have

$$(1.3) \qquad OS_k(f) = \max_{x \in \text{supp } f} OS_k(f;x) .$$

By the way we state now

Definition 1.4. For $f \in \mathcal{D}'$ we can define $OS_k(f)$ by the right hand side of (1.3) for k=1 and 2.

Remark 1.2. For $f \in \mathcal{D}'$, $OS_1(f;x_0) = -\infty$ equals to $OS_2(f;x_0) = -\infty$, which is also equivalent to saying that for any $k > 0$ there exists an open set $\Omega_k \ni x_0$ such that f is a C^k function in Ω_k. On the other hand $x_0 \notin$ sing supp f is equivalent to $OS_k(f;x) = -\infty$ in a neighbourhood of x_0.

Example. We can show easily that there exist distibutions satisfying $OS_k(f;x_0) = -\infty$ and $x_0 \in$ sing supp f at a point x_0. For example let us put $f(t) = \exp(-1/t)\sum_{n=2}^{\infty} f_n(t-\frac{1}{n})$, where $f_n(t) = \alpha(4n^2 t)t_+^n$, $\alpha(t) \in C_0^\infty$, supp $\alpha_n \subset [-1,1]$. Then $OS_k(f,0) = -\infty$ and $0 \in$ singsupp f.

Comments. As for $OS_2(f;x)$ some authors considered earlier. J.J.Duistermaat- L.Hormander [2] defined for $f \in \mathcal{D}'$

$$s_f(x) = \sup \{ t \in R;\ f \in H^t \text{ in a neighbourhood of } x \},$$

and noticed Remark 1.1 for $s_f(x)$. M.Tsuji[11] introduced the following form:

$$s.o(f;x_0) = -\lim_{\varepsilon\to 0}\sup\{OS_2(\alpha f)\ ;\ \alpha\in C_0^\infty(x_0,\varepsilon)\},$$

where $C_0^\infty(x_0,\varepsilon) = \{\alpha\in C_0;\ \mathrm{supp}\,\alpha\subset\{x;\ |x-x_0|<\varepsilon\}\}$.

From Theorem 1.1 we can verify easily $-s_f(x) = s.o(f;x) = OS_2(f;x)$.

It is remarkable that $s_f(x)$, $s.o(f;x)$ and the right hand side of (1.2) are formally determined, since the infimum of any set of real numbers always exists. While the equality (1.2) means that the infimum is approximated by an arbitrary localizing sequence in Def. 1.2. This viewpoint proceeds also to the microlocal version in the next section.

§2. Micro-local orders of singularities.

At first we introduce

Notation 2.1. For $a(x,\xi)\in S^0_{1,0} = S^0$, we note

$$[\mathrm{supp}\ a] = \text{conic supp } a = \bigcap_{m=1}^{\infty}\overline{\left\{(x,\tfrac{\xi}{|\xi|})\in R^n\times S^{n-1},\ |\xi|>m,\ (x,\xi)\in\mathrm{supp}\ a\right\}},$$

$$(\mathrm{supp}\ a) = \text{elliptic supp } a = \left\{(x,\xi')\in R^n\times S^{n-1},\ \varliminf_{r\to\infty}|a(x,r\xi')|>0\right\}.$$

Remark 2.1. (supp a) is open and [supp a] closed in $R^n\times S^{n-1}$. Evidently it holds (supp a) $\subset$ [supp a].

Definition 2.1. A sequence $\{a_n(x,\xi)\}$ of S^0 is said to be a localizing sequence to (x_0,ξ_0) if it satisfies

1) $(x_0,\xi_0)\in(\mathrm{supp}\ a_n)$ for all n ,

2) $[\mathrm{supp}\ a_n]$ converges to a point (x_0,ξ_0) in the sense that for given $\varepsilon>0$ there exists an integer N such that $[\mathrm{supp}\ a_n]$ is contained in $\{(x,\xi)\in R^n\times S^{n-1};\ |x-x_0|<\varepsilon,\ |\xi-\xi_0|<\varepsilon\}$, for all $n>N$.

Corresponding to the symbol $a(x,\xi)\in S^0$ we consider a localizer defined by the following pseudo differential operator:

$$a(D,X)u(x) = Os - \iint e^{i(x-y)\xi} a(y,\xi)u(y)\, dy đ\xi .$$

Lemma 2.1. For $u \in \mathcal{E}'$ and $a(x,\xi) \in S^0$, we have

$$\text{(2.1)} \qquad OS_k(a(D,X)u) \leq OS_k(u) \quad , \quad k=1,2.$$

Lemma 2.2. Let a_1 and a_2 belong to S^0 satisfying $[\text{supp } a_2] \subset (\text{supp } a_1)$. Then for $u \in \mathcal{E}'$ we have $OS_k(a_2 u) \leq OS_k(a_1 u)$, $k=1,2$.

Using above lemmas we can show

Theorem 2.1. Let $\{a_n(x,\xi)\}$ be an arbitrary localizing sequence to $(x_0,\xi_0) \in R^n \times S^{n-1}$. Then for $u \in \mathcal{E}'$ we have for $k = 1,2$,

$$\text{(2.2)} \qquad \lim_{n\to\infty} OS_k(a_n(D,X)u) = \inf_{a \in S^0,\ (x_0,\xi_0) \in (\text{supp } a)} OS_k(a(D,X)u) .$$

Now we describe

Definition 2.2. For $u \in \mathcal{D}'(R^n)$ the micro-local orders of singularities, $OS_k(u;x_0,\xi_0)$ at $(x_0,\xi_0) \in R^n \times S^{n-1}$, $k=1,2$, are defined by the numbers determined by (2.2).

Corresponding to Remarks 1.1 and 1.2 we have

Remark 2.2. For $u \in \mathcal{D}'$, $OS_k(u;x,\xi)$, $k=1,2$, are upper semi-continuous function on $R^n \times S^{n-1}$. Moreover we have

1) $$OS_k(u;x_0) = \max_{|\xi|=1} OS_k(u;x_0,\xi) \quad , \quad x_0 \in R^n,\ k=1,2,$$

2) $(x_0,\xi_0) \in WF\ u$ if and only if $OS_k(u;x,\xi) = -\infty$ holds in a neighbourhood of (x_0,ξ_0) in $R^n \times S^{n-1}$, k = 1or 2.

Now we consider $OS_k(u;x,\xi)$ more precisely.

Definition 2.3. Let A be an arbitrary set on $R^n \times S^{n-1}$. Then we define orders of singularities on A as follows.

$$\mathrm{OS}_k(u;A) = \sup_{(x,\xi)\in A} \mathrm{OS}_k(u;x,\xi) . \tag{2.3}$$

Then we can verify

Lemma 2.3. Let $a(x,\xi)$ belong to S^0. Then for $u\in\mathcal{D}'$ it holds

$$\mathrm{OS}_k(u;\ (\mathrm{supp}\ a)) \leq \mathrm{OS}_k(a(D,X)u) \leq \mathrm{OS}_k(u;[\mathrm{supp}\ u]),\quad k=1,2. \tag{2.4}$$

and the following

Theorem 2.2. Suppose $u\in\mathcal{D}'$ and $(x_0,\xi_0)\in R^n\times S^{n-1}$. Then for any given $\varepsilon>0$, there exists a positive constant such that

$$\mathrm{OS}_k(u;x_0,\xi_0) \leq \mathrm{OS}_k(a(D,X)u) \leq \mathrm{OS}_k(u;x_0,\xi_0)+\varepsilon\quad ,\ k=1,2. \tag{2.5}$$

for all $a(x,\xi)\in S^0$ satisfying $(x_0,\xi_0)\in(\mathrm{supp}\ a)$ and $[\mathrm{supp}\ a]\subset U_\delta(x_0,\xi_0)$ $=\left\{(x,\xi)\in R^n\times S^{n-1};\ |x-x_0|<\delta\ ,|\xi-\xi_0|<\delta\right\}$. Moreover (2.5) holds even if we replace $a(D,X)$ by $a(X,D)$ the usual form of pseudo-differential operator with symbol $a(\xi,x)$ for $u\in\mathcal{E}'$.

From Theorem 2.2 we can see that the localizing process by $\{a_n\}$ is concerned essentially in $[\mathrm{supp}\ a_n]$ and $(\mathrm{supp}\ a_n)$. Hence it is useful to choose special localizing sequences.

Example 2.1. Let $\tilde{\beta}(x)\in C_0^\infty$; $=1$ on $\left\{x;\ |x|\leq\frac{1}{2}\right\}$, and $=0$ on $\left\{x;\ |x|\geq 1\right\}$. Put $\tilde{\alpha}(\xi)=\tilde{\beta}(\xi)$. For $(x_0,\xi_0)\in R^n\times S^{n-1}$ we put

$$\beta_n(x) = \beta(n(x-x_0)),\qquad \alpha_n(\xi) = (n(\frac{\xi}{|\xi|}-\xi_0))\,\chi(\xi) \tag{2.6}$$

where $\chi(\xi)\in C^\infty$; $=1$ for $|\xi|\geq 1$, $=0$ for $|\xi|\leq\frac{1}{2}$. Then $\left\{\alpha_n(\xi)\,\beta_n(\xi)\right\}$ is a localizing sequence to (x_0,ξ_0).

Example 2.2. Here we point out another localizing sequence relating to the micro-localization used by Mizohata [8] and [9]. for $(x_0,\xi_0)\in R^n\times S^{n-1}$ we note simply using $\tilde{\beta}(x)$ and $\tilde{\alpha}(\xi)$

$$(2.7) \qquad \beta(x) = \tilde{\beta}\left(\frac{x-x_o}{r_o}\right), \qquad \alpha_n(\xi) = \tilde{\alpha}\left(\frac{\xi/n - \xi_o}{r_o}\right), \qquad 0 < r_o < 1.$$

We can verify that $\left\{\Sigma \frac{1}{n} \alpha_n(\xi)\beta(x)\right\}_{r_o \in (0,1)}$ and $\left\{\frac{\alpha_n(\xi)\beta(x)}{(1+|\xi|^2)^{1/2}}\right\}_{r_o \in (0,1)}$ are localizing sequences to (x_o, ξ_o) as r_o tends to 0, which we consider again in §4.

Comments. J.J.Duistermaat- L.Hormander [2] introduced the following definition : For $u \in \mathcal{D}'$,

$$s_u^*(x,\xi) = \sup\{t;\ u \in H^t \text{ at } (x,\xi)\},$$

where $u \in H^t$ at (x,ξ) means that $u = u_1 + u_2$ with $u_1 \in H^t$ and $(x,\xi) \notin WF(u)$ and Remark 2.2 for $-s_u^*(x,\xi)$ was given there. Using Theorem 2.1 we can verify $-s_u(x,\xi) = OS_2(u;x,\xi)$ and recognize the set of localization given by $\{a(D,X)u;\ a \in S^0,\ (\text{supp } a) \ni (x_o, \xi_o)\}$ is sufficiently universal. Thus it suffices to consider localizing sequences as in Example 2.1 and 2.2. This makes our way of thinking simple and turns out to be useful when we consider concrete calculus for some applications.

§3. Applications.

Here using micro-local order of singularities we state some results concerning hypoellipticity and propagations of singularities. This would indicate us that above both properties are not different from viewpoints of microlocal orders of singularities.

At first from Theorems 2.1 and 2.2, it follows

Corollary 3.1. Let $a(x,\xi) \in S^m$. Suppose $(x_o, \xi_o) \in (\text{supp } a/(1+|\xi|^2)^{\frac{m}{2}}$. Then we have for $u \in \mathcal{E}'$,

$$(3.1) \qquad OS_k(a(X,D)u;x_o,\xi_o) = OS_k(u;x_o,\xi_o) + m, \quad k=1,2.$$

Remark 3.1. If $(x_0, \xi_0) \in (\mathrm{supp}\ a/(1+|\xi|^2)^{m/2})$ for $a \in S^m$, then we can see that $a(X,D)u = f$ is micro-locally hypoelliptic in a neighbourhood of (x_0, ξ_0). Hence , if we want to consider the propagations of singularities under the assumption that f is smooth, it suffices to confine ourselves to considering on $\{(x,\xi) \in R^n \times S^{n-1};\ a_m(x,\xi) = 0\}$, where a_m is the principal part of a. This argument continues to the next theorem.

Theorem 3.1. Suppose $p(x,\eta) \in S^m$. Let $\phi(x,\eta) \in S^1$ be a real phase function and homogeneous in η for large $|\eta|$. Assume that $J = \phi(x,\eta) - x\cdot\eta$ satisfies

$$(3.2) \qquad \sum_{|\alpha|+|\beta| \leq 2} \sup_{|\eta|>1} \left\{ J^{(\alpha)}_{(\beta)}(x,\eta)\ (1+|\eta|^2)^{|\alpha|/2} \right\} < 1.$$

Then we have for Fourier integral operator in R^n with amplitude $p(x,\eta)$ and phase function ϕ ,

$$(3.3) \qquad OS_1(P\ u;\ x,\xi) \leq OS_1(u;y,\eta) + m + \frac{n}{2} \quad ,$$

$$(3.4) \qquad OS_2(P\ u;\ x,\xi) \leq OS_2(u;y,\eta) + m \ ,$$

where (x,ξ) and (y,η) are related by $y = \phi_\eta(x,\eta)$ and $\xi = \phi_x(x,\eta)$.

Problem. We can expect that $\frac{n}{2}$ will be modified , though its caliculation is not simple.

Comments. Hörmander[3] mentioned to (3.1) for k = 1, as a limit of micro-local hypoellipticity. Conversely it is possible to understand that micro-local hypoellipticity is a consequence of some relation micro-local order of u and au. At that time we can tray to take some suitable localizing sequences. The author was interested in OS_1 at first in order to consider propagations of singularities and supports of solutions to simple hyperbolic mixed problems in [6]. The notion of orders of sigularities will play impotant role when we consider solutions of mixed problems more cocretely. As for energy estimates there exists a certain difference between non-homogeneous Dirichlet and Neumann problems for second order hyperbolic equations, (cf. [7]). This difference should appear also in propagation phanomena of orders of singularities. Finally notations of Theorem 3.1 are cited from H.Kumano-go [5].

§4.Mizohata's localization and orders of sigularities.

At first let us recall Theorem 2.2 and Example 2.2. Then we have

$$(4.1)\qquad OS_k(u;U_{r_0/2}(x_0,\xi_0)) \leq OS_k(\Sigma\,\tfrac{1}{n}\alpha_n(D)\beta u)$$
$$= OS_k(\Sigma\,(1-\Delta)^{1/2}\alpha_n(D)\beta u) \leq OS_k(u;U_{r_0}(x_0,\xi_0)).$$

Now we describe as in [10]

Notation. For $u\in\mathcal{D}'$, we say $u\in H^t_{(x_0,\xi_0)}$, if, for some $r_0>0$,

$$\int_{|\xi/|\xi|-\xi_0|\leq r_0/2} (1+|\xi|)^{2t}|\widehat{\beta u}(\xi)|^2\,d\xi < \infty ,$$

where β is a function defined in (2.7). This definition is equivalent to that in M.Bony [1], where pseido-differential operators are used as localizers.

Now we can state

Theorem 4.1. Let $u\in\mathcal{D}'$, and $(x_0,\xi_0)\in R^n\times S^{n-1}$. Then

$$(1)\qquad OS_2(u;x_0,\xi_0) = -s ,$$

is equivalent to saying that, for any $\varepsilon>0$, there exists $r_0>0$ such that one of the following (2) $\sim$ (6) holds :

$$(2)\qquad OS_2(\Sigma\,\tfrac{1}{n}\alpha_n\beta u) < -(s-\varepsilon),$$

$$(3)\qquad u\in H^{s-\varepsilon}_{(x_0,\xi_0)} ,\qquad (4)\qquad \|\alpha_n\beta u\| = O(n^{-(s-\varepsilon)}) ,$$

$$(5)\quad \Sigma\,\|\alpha_n\beta u\|^2\, n^{2(s-\varepsilon)-1} < \infty ,\qquad (6)\,\Sigma\|\alpha_n\beta u\|\, n^{s-\varepsilon-1} < \infty .$$

Here we remark only that we have (1)$\Leftrightarrow$(2)$\Leftarrow$(3) from Theorem 2.2 and Example 2.2, (3)$\Leftrightarrow$(5) from [10], (4)$\Rightarrow$(3) from Y.Takei [12] and (2)$\Rightarrow$(4)$\Rightarrow$(6)$\Rightarrow$(2) by direct caliculus.

Theorem 4.2. We have Theorem 4.1 replaced OS_2 by OS_1, if $H^{s-\varepsilon}_{(x_0,\xi_0)}$ and $\|\alpha_n \beta u\|$ are replaced respectively by

$$\mathcal{B}^{s-\varepsilon}_{(x_0,\xi_0)} = \left\{u;\ \sup_{|\xi_0 - \xi/|\xi||<r_0} (1+|\xi|)^{s-\varepsilon}|\widehat{\beta u}(\xi)| < \infty\right\},$$

$$\|\alpha_n \beta u\|_1 = \sup_{\xi\in R^n}|\alpha_n(\xi)\widehat{\beta u}(\xi)| .$$

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Applications of two-timing methods to heterogeneous reaction-diffusion systems

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There are so many mathematical models characterized by quasi-linear parabolic systems of differential equations which describe actual problems arising in physics, chemistry, biology and many other fields. When such systems take of the form

$$(1) \qquad \frac{\partial u}{\partial t} = \mathrm{div}\Big(D(x,t,u)\nabla u + E(x,t,u)\Big) + f(x,t,u),$$

$u = (u_1,\cdots,u_n)$, in which f may be replaced by $f(x,t,u,\nabla u)$, they are often called *reaction-diffusion-advection systems*. Here, $D(x,t)$ is a nonnegative definite matrix. If D is a constant matrix , f is independent of t and x and $E \equiv 0$, (1) is reduced to *homogeneous* reaction-diffusion systems

$$(2) \qquad \frac{\partial u}{\partial t} = D\Delta u + f(u).$$

In most applications, D is diagonal. (2) are extensively investigated by numerous authors from both analytical and numerical view points. In this paper, we shall not touch upon them but refer to excellent reviews by Fife[5] and Smoller[23].

On the other hand, we encounter *spatially inhomogeneous* or *heterogeneous* reaction-diffusion-advection systems such as

$$(3)\qquad \frac{\partial u_i}{\partial t} = \mathrm{div}\left(d_i(x)\nabla u_i + u_i \nabla e_i(x)\right) + f_i(x,u) \quad (i = 1,\ldots,n),$$

in which $f_i(x,u)$ and $d_i(x)$, $e_i(x)$ explicitly depend on space variables x. These models occur widely as ones for dynamics of chemical substances or biological species in heterogeneous media or environments(Okubo[17], Fife[5], etc.). Let us show one simple but very suggestive model equation introduced by Fisher[7] in population genetics though there are many other models described by heterogeneous reaction-diffusion equations(Gurney and Nisbet[9], Kawasaki and Teramoto[11], Kurland[12], Nagilaki[16], Pacala and Roughgarden[18], Skellam[22] and their references therein). It is described by

$$(4)\qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda m(x)u(1 - u), \; x \in I \mathrel{\overset{\leftarrow}{=}} (0,\ 1),\; t > 0$$

with Neumann baundary conditions. Since the variable u in (4) denotes the frequency of a gene, we pay attention to only solutions of (4) for which $0 \le u \le 1$. Assume $\int_I m(x)dx = 1$. When $m(x)$ is constant, that is, $m(x) \equiv 1$, we know that any solution $u(t,x)$ satisfying $0 \le u \le 1$, and $u \not\equiv 0$ tends to 1 as $t \to \infty$ for any $\lambda > 0$. On the other hand, when $m(x)$ is not constant and $m(x) < 0$ on a set of positive measure, Fleming[8] showed that there exists $\lambda_0 > 0$ such that for $0 < \lambda \le \lambda_0$, $u(t,x)$ satisfying $0 \le u \le 1$ and $u \not\equiv 0$ tends to 1 as $t \to \infty$, while for $\lambda > \lambda_0$, there appears a unique stable inhomogeneous equilibrium $\phi_\lambda(x)$ of (4),

$0 < \phi_\lambda(x) < 1$ on $\overline{I}$ and $u(t,x)$ satisfying $0 \leq u \leq 1$, and $u \not\equiv 0$, $u \not\equiv 1$ tends to $\phi_\lambda(x)$ as $t \to \infty$. Thus, the appearance of heterogeneity of $m(x)$ drastically changes the situation. For the study of such heterogeneous systems, there have been at least three approaches. The first two, which are essentially powerful to scalar equations, are the super- and sub-solutions methods (Fife and Peletie[6], Howes[20], Leung and Bendjilali[13] , Matano[14], Pauwelussen and Peletier[19]) and the variational ones (Fleming[8], Kawasaki and Teramoto[11], Yanagida[25]), which give the existence and sometimes the stabilities of inhomogeneous stationary solutions. A second is perturbed bifurcation techniques, which can widely applied to homogeneous reaction-diffusion systems of equations perturbed by weak heterogeneities (e.g. Mimura and Nishiura[15]). Although we know these three approaches, it is still hard to study spatial and/or temporal pattern formation in heterogeneous reaction-diffusion systems of equations. For this purpose, Su Yu[24] considered a fairly general class of systems

$$(5) \quad \begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + \varepsilon f(x,\varepsilon t,u), \quad t > 0, \ x \in \Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

with Neumann boundary conditions. Here, Ω is a bounded domain in R^m. Under certain assumptions, he showed that when ε is sufficiently small, there exist $K > 0$ and $T(\varepsilon) > 0$ such that

$$\|u(t,*) - \bar{u}(\varepsilon t)\|_{L^2(\Omega)} \leq K\varepsilon$$

for $t \geq T(\varepsilon)$, where $u(t,x)$ is the solution of (5) and $\bar{u}(\varepsilon t)$ is the solution of

$$(6)\qquad \begin{cases} \dfrac{d\bar{u}}{dT} = \bar{f}(T,\bar{u}), \quad T > 0, \\ \bar{u}(0) = \bar{u}_0 , \end{cases}$$

with $\bar{u}_0 \fallingdotseq (\text{meas. } \Omega)^{-1}\int_\Omega u_0(x)dx$ and $\bar{f}(T,\bar{u}) \fallingdotseq (\text{meas. } \Omega)^{-1}\int_\Omega f(x,T,\bar{u})dx$. Thus, we can study the effect of heterogeneity of the reaction term f on the asymptotic behavior of solutions of (5) through the analysis of O.D.E. (6), which is much easier than that of P.D.E. Unfortunately, his result did not argue the transient behavior of solutions for $0 < t \leq T(\varepsilon)$. Furthermore if we are interested in dynamics of populations in heterogeneous environments for instance, we meet heterogeneous reaction-diffusion-advection systems of (3) rather than (5) (Okubo[17], Comins and Blatt[1], Fife and Peletier[6], Shigesada et.al[21], Yanagida[25], Howes[10]).

For such systems, Shigesada[20], Ei and Mimura[4] analyzed the transient as well as asymptotic behavior of solutions for ecological model equations of the form

$$
(7) \quad \begin{cases} \dfrac{\partial u_i}{\partial t} = \mathrm{div}(d_i \nabla u_i + u_i \nabla e_i(x)) + \varepsilon f_i(x,u), \quad x \in \Omega \subset R^m, \ t > 0, \\ u_i(0,x) = u_{0i}(x), \quad x \in \Omega, \quad u = (u_1, \cdots, u_n) \quad (i = 1, \ldots, n) \end{cases}
$$

with a positive small parameter ε. Here, Ω is a bounded domain in R^m and the boundary conditions are no-flux ones and $d_i > 0$ ($i = 1, \ldots, n$). To investigate the spatial and/or temporal pattern formation of solutions of (7), Shigesada[20] applied the two-timing method to (7) and constructed the lowest-order approximate solution of the form

$$
(8) \quad u_i^0(t, \varepsilon t, x) = N_i(\varepsilon t)\varphi_i(x) + w_i(t,x) \quad (i = 1, \ldots, n),
$$

where $\varphi_i(x) \fallingdotseq \exp\{-e_i(x)/d_i\} / \int_\Omega \exp\{-e_i(x)/d_i\}dx$ and $N_i(T)$ is the solution of

$$
(9) \quad \begin{cases} \dfrac{dN_i}{dT} = F_i(N_1, \cdots, N_n) \fallingdotseq \displaystyle\int_\Omega f_i(x, N_1\varphi_1(x), \cdots, N_n\varphi_n(x))dx, \\ N_i(0) = \displaystyle\int_\Omega u_{i0}(x)dx \quad (i = 1, \ldots, n) \end{cases}
$$

and $w_i(t,x)$ is the solution of

$$
(10) \quad \begin{cases} \dfrac{\partial w_i}{\partial t} = \mathrm{div}(d_i \nabla w_i + w_i \nabla e_i(x)), \\ w_i(0,x) = u_{0i}(x) - \displaystyle\int_\Omega u_{0i}(x)dx \cdot \varphi_i(x) \quad (i = 1, \ldots, n), \end{cases}
$$

with the no-flux boundary conditions. Ei and Mimura[4] proved that if the solution of (9) converges to an asymptotically stable equilibrium of (9), then the solution $u_i(t,x;\varepsilon)$ of (7), satisfies

$$\| u_i(t,*;\varepsilon) - u_i^0(t,\varepsilon t,*) \|_{L^\infty(\Omega)} \le O(\varepsilon),$$

specially,

$$\left| \int_\Omega u_i(t,x;\varepsilon)dx - N_i(\varepsilon t) \right| \le O(\varepsilon)$$

uniformly on $t \in [0, \infty)$ for $i = 1,\ldots,n$.

On the other hand, it is possible for the equations of (9) to exhibit stable limit cycles or strange attractors such as chaos. Especially, in the case of limit cycles, it is observed by computer simulations that $u(t,x;\varepsilon)$ seems to be periodic and the orbit described by $(\int_\Omega u_1(t,x;\varepsilon)dx,\cdots,\int_\Omega u_n(t,x;\varepsilon)dx)$ agrees fairly well with the orbit $(N_1(\varepsilon t),\cdots,N_n(\varepsilon t))$ in phase plane R^n though Ei and Mimura[4] have not discussed this case.

This observation motivates us to study the case that the (9) posses stable limit cycles. Our result is as follows: Let γ be a stable limit cycle of (9) and assume that $(N_1(\varepsilon t),\cdots,N_n(\varepsilon t)) \to \gamma$ as $t \to \infty$. Then there exist $C > 0$ and $t_\varepsilon > 0$ such that

$$\mathrm{dist}\left\{\gamma, \left(\int_\Omega u_1(t,x;\varepsilon)dx,\cdots,\int_\Omega u_n(t,x;\varepsilon)dx\right)\right\} \le C\varepsilon$$

for any $t \ge t_\varepsilon$ when ε is sufficiently small. In [3], we will analyze (7) in detail with applications to more concrete actual problems of two-species prey-predator model and investigate quantitatively the effect of heterogeneity.

There is another application. Let us consider the following systems:

$$(11) \qquad \frac{\partial u_i}{\partial t} = d_i \Delta u_i + \varepsilon \{ \Sigma_{j=1}^{m} a_{ij}(x,u) \frac{\partial u_i}{\partial x_j} + f_i(u) \}, \ x \in \Omega, \ t > 0$$

$$(i = 1,...,n)$$

with Neumann boundary conditions and

$$(12) \qquad u_i(0,x) = u_{i0}(x), \ x \in \Omega,$$

where $u = (u_1,\cdots,u_n)$, Ω is a bounded domain in R^m and $d_i > 0$. Conway, Hoff and Smoller[2] proved that when ε is sufficiently small and (11) admit a compact positively invariant set $\Sigma \subset R^n$ independent of small ε, then any solution $(u_1(t,x),\cdots,u_n(t,x))$ of (11), (12) with values in Σ converges uniformly and exponentially to their spatial averages $\bar{u}_i = (\text{meas. } \Omega)^{-1}\int_\Omega u_i(t,x)dx$ (i = 1, ... ,n), where $\bar{u}_i(t)$ satisfies

$$(13) \qquad \frac{d\bar{u}_i}{dt} = \varepsilon f_i(\bar{u}_1,\cdots,\bar{u}_n) + O(\varepsilon e^{-\sigma t}) \ \text{as} \ t \to \infty \quad (i = 1,...,n)$$

for some $\sigma > 0$. If we apply our results to (11), (12), the above statement is valid without assuming the existence of such an invariant set Σ. Details will be stated in [3].

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Singular Limits for the Compressible Euler Equation in an Exterior Domain, II——Bodies in a Uniform Flow

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1. Formulation of the problem. Suppose an unbounded domain Ω in R^3 exterior to a bounded obstacle with compact smooth boundary S is occupied by an ideal gas. Let P be its pressure and V the velocity. The entropy is assumed to be constant. Then the compressible Euler equation in a suitable non-dimensional form is written as

$$\begin{aligned} &\partial_t P + (V\cdot\nabla)P + \gamma P\nabla\cdot V = 0, \\ &\partial_t V + (V\cdot\nabla)V + \lambda^2 P^{-1/\gamma}\nabla P = 0, \\ &\langle n, V\rangle = 0 \quad \text{on} \quad S, \end{aligned} \tag{1.1}$$

where γ is a constant > 1, n is the outer unit normal to S and λ is a large parameter propotional to the inverse of the Mach number. Let $H^m = H^m(\Omega)$ be the usual Sobolev space of order m. In our previous works [1], [2], we have already shown that the solution of the above equation converges to that of the incompressible Euler equation as $\lambda \to \infty$ under the main condition that the initial pressure $P^\lambda(0) = \text{Const.} + O(\lambda^{-1})$ and the initial velocity $V^\lambda(0) \in H^{N+1}$, $N \geq 4$. The assumption that the initial velocity belongs to $L^2(\Omega)$ is rather restrictive, since it excludes physically important flows which are both solenoidal and irrotational. In fact, if a vector field $V(x) \in L^2(\Omega)$ satisfies $\mathrm{div}\, V = 0$, $\mathrm{curl}\, V = 0$ and the boundary condition, it must be identically equal to 0. In this note, we consider the flow constant at infinity as an important example of such a non L^2 flow.

Let a constant non zero vector $\xi \in R^3$ be fixed, which is the velocity at infinity. Take $w_0(x) \in \mathscr{B}(\Omega)$ such that

$$
\begin{aligned}
&\mathrm{div}\, w_0 = 0 \quad \text{in} \quad \Omega, \\
&\langle n, w_0 \rangle = 0 \quad \text{on} \quad S, \\
&|\partial^\alpha (w_0(x) - \xi)| \le C\langle x \rangle^{-\varepsilon - |\alpha|}, \quad |\alpha| = 0,1, \qquad (1.2) \\
&C \text{ and } \varepsilon \text{ being positive constants}, \\
&(w_0 \cdot \nabla) w_0 \in L^2(\Omega).
\end{aligned}
$$

For instance, one can take $w_0(x) = \xi + \nabla\varphi(x)$, where φ solves the following Neumann problem :

$$
\begin{aligned}
&\Delta\varphi = 0 \quad \text{in} \quad \Omega, \\
&\frac{\partial\varphi}{\partial n} = - \langle n, \xi \rangle \quad \text{on} \quad S.
\end{aligned}
$$

Indeed, $\varphi(x)$ is written by a single layer potential on the boundary

$$\varphi(x) = \int_S \mu(y) \frac{1}{|x-y|} dS_y,$$

whence $\qquad |\partial^\alpha (w_0(x) - \xi)| \le C_\alpha \langle x \rangle^{-2-|\alpha|}.$

One can also construct other examples by setting $w_0(x) = \xi +$ $\mathrm{curl}\, A(x)$.

In order to treat (1.1), we transform P into $Q = \frac{\gamma}{\gamma-1} P^{1-1/\gamma}$. Then (1.1) is rewritten as

$$
\begin{aligned}
&\partial_t Q + (V\cdot\nabla)Q + (\gamma-1)Q\nabla\cdot V = 0, \\
&\partial_t V + (V\cdot\nabla)V + \lambda^2 \nabla P = 0.
\end{aligned}
$$

We set $\gamma = 2$ for the sake of simplicity. We are going to assume that the initial pressure behaves like Const. + $O(\lambda^{-1})$. Thus, we set, without loss of generality, $Q = 1 + p/\lambda$. Then we arrive at

$$\partial_t p + (V\cdot\nabla)p + p\nabla\cdot V + \lambda\nabla\cdot V = 0,$$
$$\partial_t V + (V\cdot\nabla)V + \qquad \lambda\nabla p = 0.$$

Since we consider the velocity close to w_0 specified by (1.2), we set $V = v + w_0$ and obtain the following equation

$$\begin{aligned}&\partial_t p + (v\cdot\nabla)p + p\nabla\cdot v + (w_0\cdot\nabla)p + \lambda\nabla\cdot v = 0,\\ &\partial_t v + (v\cdot\nabla)v + (v\cdot\nabla)w_0 + (w_0\cdot\nabla)v + \lambda\nabla p = -(w_0\cdot\nabla)w_0,\\ &\langle v,n\rangle = 0 \quad \text{on} \quad S.\end{aligned} \tag{1.3}$$

2. Main results. The following assumptions are imposed on the initial data p_0^λ, v_0^λ :

(A-1) $\{(p_0^\lambda, v_0^\lambda);\lambda>0\}$ *is a bounded set in* $H^{N+1}(\Omega)\cap L^1(\Omega)$, *where* N *is an integer* ≥ 4.

(A-2) *The compatibility condition is satisfied up to order* $N+1$.

(A-3) $P_S v_0^\lambda \to v_0^\infty$ *in* $H^N(\Omega)$ *as* $\lambda \to \infty$, P_S *being the projection onto the solenoidal fields.*

Let $\|\cdot\|_m$ be the norm of H^m. Our main results are the following theorems.

THEOREM A. *There exist constants* $T > 0$ *and* $\Lambda > 0$ *such that for any* $\lambda > \Lambda$, *there exists a unique solution* $p^\lambda(t)$, $v^\lambda(t) \in \bigcap_{k=0}^{N} C^k(I;H^{N-k}(\Omega))$, $I = [0,T]$, *of the above equation* (1.3). *Moreover, it satisfies the following uniform estimate*

$$\sup_{\lambda>\Lambda,\, t\in I} (\|p^\lambda(t)\|_N + \|v^\lambda(t)\|_N) < \infty.$$

THEOREM B. *For* $0 < t \leq T$, $p^{\lambda}(t) \to 0$ *and* $v^{\lambda}(t) \to v^{\infty}(t)$ *in* $H^{N-1}_{loc}(\bar{\Omega})$ *as* $\lambda \to \infty$. *Furthermore,* $u^{\infty}(t) = v^{\infty}(t) + w_0$ *satisfies the following incompressible Euler equatrion*

$$\partial_t u^{\infty} + (u^{\infty}\cdot\nabla)u^{\infty} + \nabla q^{\infty} = 0, \quad \textit{in} \ \ \Omega, \quad t \in I,$$
$$\operatorname{div} u^{\infty} = 0, \quad \textit{in} \ \ \Omega, \quad t \in I,$$
$$\langle n, u^{\infty}\rangle = 0 \quad \textit{on} \ \ S,$$
$$u^{\infty}(0) = v_0^{\infty} + w_0,$$
$$u^{\infty}(t) - w_0 \in H^{N-1}(\Omega),$$

where $q^{\infty} = q^{\infty}(t) \in H^{N-1}_{loc}(\bar{\Omega})$ *is calculated from* $u^{\infty}(t)$.

3. Methods of the proof. The proof of Theorem A is almost the same as in [1], §5. Letting $f^{\lambda} = {}^t(p^{\lambda}, v^{\lambda})$ and L be the linearized operator of acoustics

$$L = -i\begin{pmatrix} 0 & \nabla \\ {}^t\nabla & 0 \end{pmatrix} \quad \text{in} \ \ L^2(\Omega)$$

with the boundary condition $\langle v, n\rangle = 0$ on S, we rewrite (1.3) as follows

$$(1.4) \qquad \partial_t f + A(f)f + (w_0\cdot\nabla)f + i\lambda Lf = F,$$

where $F = {}^t(0, -(w_0\cdot\nabla)w_0)$ and $A(\cdot)$ be the differential operator defined by

$$A(g)f = {}^t((w\cdot\nabla)p + q\nabla\cdot v,\ (w\cdot\nabla)v + (v\cdot\nabla)w_0)$$

for $f = {}^t(p, v)$, $g = {}^t(q, w)$. The linearized equation for (1.4) is

$$(1.5) \qquad \partial_t f + A(g)f + (w_0\cdot\nabla)f + i\lambda Lf = G.$$

Let Γ_0 be the projection onto the null space of L and $\Gamma = 1 - \Gamma_0$. For $f = {}^t(p, v)$, $\Gamma_0 f = {}^t(0, P_S v)$. We split the solution f

of (1.5) into two parts : $f = \Gamma_0 f + \Gamma f$. $\Gamma_0 f$ satisfies the incompressible Euler equation which can be treated separately (see e.g. [3]). To estimate Γf we use the coercivenes estimate :

(1.6) *If* $f \in D(L)$ *and is orthogonal to the null space of* L,

$$\|f\|_{m+1} \leq C_m(\|f\| + \|Lf\|_m), \quad m \geq 0.$$

These two estimates enable us to derive the following energy inequality : Let

$$\|f(t)\|_{X^m} = \sum_{k=0}^{m-1} \|(\tfrac{1}{\lambda}\partial_t)^k f(t)\|_{m-k}, \quad m \geq 1$$

and $\gamma = \sup_{t\in I} (\|g(t)\|_{X^{N+1}} + \|w_0\|_{\mathscr{B}^{N+1}})$. Then we have

$$\begin{aligned}(1.7) \qquad \|f(t)\|_{X^m} \leq C\, e^{tC(\gamma)}(\|f(0)\|_{X^m} &+ \int_0^t \|G(s)\|_{X^m} ds \\ &+ \frac{1}{\lambda}\|G(t)\|_{X^m}),\end{aligned}$$

$$1 \leq m \leq N+1, \quad t \in I.$$

Theorem A easily follows from (1.7) by the standard method of iteration.

To prove Theorem B, we study an asymptotic property as $\lambda \to \infty$ of the equation

$$(1.8) \qquad \begin{aligned} &\partial_t p + (w_0\cdot\nabla)p + \lambda\nabla\cdot v = 0, \\ &\partial_t v + (w_0\cdot\nabla)v + \lambda\nabla p = 0, \\ &\langle v,n\rangle = 0 \quad \text{on} \quad S. \end{aligned}$$

We construct a 1-parameter group of diffeomorphism $\varphi_t(x)$ by

$$\frac{d}{dt}\varphi_t(x) = w_0(\varphi_t(x)), \quad \varphi_0(x) = x,$$

and define $(\Phi(t)f)(x) = f(\varphi_t(x))$. Let $U_\lambda(t)$ be the unitary group for (1.8). The important fact is the following lemma.

LEMMA. $U_\lambda(t)\Gamma - \Phi(-t)e^{-i\lambda tL}\Gamma \to 0$ *strongly in* $L^2(\Omega)$ *as* $\lambda \to \infty$ *for any* $t \in R$.

Thanks to the above lemma, one can reduce the asymptotic properties of $U_\lambda(t)\Gamma$ to those of $e^{-it\lambda L}\Gamma$ which have been studied in [1].

4. Remaining problems. So far we have studied the equation (1.1) in an exterior domain, many problems are left unsolved. For instance, it is not easy to relate the pressure $q^\infty(t)$ to the original $p^\lambda(t)$. Even in the case $w_0(x) = 0$, some geometric assumptions on the boundary (e.g. non trapping condititions) seem to be necessary. If we consider the non-isentropic fluid, in order to prove Theorem B, we have to study spectral and scattering problems of the linearized operator of acoustics with coefficients depending on time, which seems to be a very subtle problem. Theorem A also holds for the interior domain (we take $w_0 = 0$). In this case, the incompressible limit (Theorem B) is derived under the additional assumption that, roughly, $p_0^\lambda = O(\lambda^{-1})$, div $v_0^\lambda = O(\lambda^{-1})$, and the initial layer does not appear. But what occurs when we consider the limit $\lambda \to \infty$ under our original assumptions ? The incompressible limit for the boundary value problem of the compressible Navier-Stokes equation is also an interesting problem. For the stationary case, this was studied by [4]. But the non stationary problem is yet unsolved. A good explanation of these problems of singular limits in non-linear equations of fluid is given in [5], Chapter 2.

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Spectral analysis for linearized MHD operators

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The linearized MHD equation, around the equilibrium quantities $\{\rho, P, B\}$ which satisfy:

(0.1) grad $P = j \times B$, $j=$ rot B, div $B = 0$, $\rho \geq \rho_0 > 0$:arbitrary, $P \geq P_0 > 0$,

is written as:

$$(0.2) \quad \rho\frac{\partial^2 \xi}{\partial t^2} = -K\xi \equiv \mathrm{grad}(\gamma P(\mathrm{div}\ \xi)+(\mathrm{grad}\ P)\xi) + B\times\mathrm{rot}(\mathrm{rot}(B\times\xi))-(\mathrm{rot}\ B)\times\mathrm{rot}(B\times\xi).$$

Here, $\xi:\Omega\subset R^3 \to R^3$ is the Lagrangian displacement vector in the vicinity of the equiliblium and γ is a positive constant.

§1. Some special equilibria

The general solution of the equation (0.1) is not known, while if the domain Ω is of special shape, we can construct some equilibria with certain symmetry. The simplest example is a constant solution in the whole space R^3. We consider the cases where the domains are bounded. In these cases, the simplest one would be the flat torus, the next a periodic cylinder and an axially symmetric region. When the domain is cylindrically or axially symmetric, there is a master equation called Grad-Shafranoff equation:

(1.1) $\quad -\Delta\psi=F(\psi)$

where ψ is a stream function for the magnetic field B. In the cylindrical case, $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$ and $F(\psi)=(\partial/\partial\psi)(P+(1/2)I^2)$ with an arbitrary positive definite function P of ψ and a function I of ψ. In the case of axial symmetricity, $\Delta=r(\partial/\partial r)(1/r)(\partial/\partial r)+\partial^2/\partial z^2$ and $F(\psi)=r^2(\partial P/\partial\psi)+I(\partial I/\partial\psi)$ with the same type of functions

P and I. In these cases, the equilibria are given by

(1.2) $P=P,\ B=(\partial\psi/\partial y,-\partial\psi/\partial x,I)$ (cylindrical case)

and

(1.3) $P=P,\ B=((1/r)(\partial\psi/\partial z),(1/r)I,-(1/r)(\partial\psi/\partial r))$

(axially symmetric case).

Further, as a more special case in the flat torus, there is an equilibrium:

(1.4) $P=P(x),\ B=(0,b(x)\sin\phi(x),b(x)\cos\phi(x))$

where

(1.5) $(P+b^2/2)'=0$, $'=\partial/\partial x$

and P,b and ϕ are periodic functions of x with $P(x)\geqq P_0>0$

In the circular cylinder case,

(1.6) $P=P(r),\ B=(0,b(r)\sin\phi(r),b(r)\cos\phi(r))$

where

(1.7) $(P+b^2/2)'+b^2\sin^2\phi/r=0,\ '=\partial/\partial r$

with smooth functions P,b and ϕ of r satisfying:$\phi(0)=b(0)'=0$

§2. Resolvent of linearized MHD operators

To establish the operator K, we adopt the method to construct the resolvent of the operator firstly. In the case of §1, by appropriate unitary transformations caused from change of variables, change of weight and change of moving coordinates, the operator K is written as

(2.1) $K=\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$

First, we calculate the formal inverse operator of K. For example, in the case of circular cylinder with radius r_0, A,B,C are, after Fourier decomposition: $\xi=\exp(ikz+im\theta)\mu$,

(2.2) $A=-\partial\{(b^2+\gamma P)/r^2\}\partial+\{b^2\sin^2\phi/r^2\}'+b^2k_\phi^2$

(2.3) $B=(-i\partial\{(b^2+\gamma P)m_\phi/r\}+2ib^2k(\sin\phi)/r,\ -i\partial\{\gamma Pk_\phi/r\})$

(2.4) B^*: formal adjoint of B

$$(2.5)\quad C=\begin{pmatrix} m_\phi^2(b^2+\gamma P)+b^2k_\phi^2 & m_\phi k_\phi\gamma P \\ m_\phi k_\phi\gamma P & k_\phi^2\gamma P \end{pmatrix}$$

with $m_\phi=k(\cos\phi)+(m/r)(\sin\phi)$, $k_\phi=(m/r)(\cos\phi)-k(\sin\phi)$

and $\partial='=d/ds$ where $s=\log r$, $-\infty<s\leqq s_0=\log r_0$.

The resolvent $(K+\lambda\rho)^{-1}$, $\lambda>>0$, is then calculated as

$$(2.6)\quad (K+\lambda\rho)^{-1}=\begin{pmatrix} E_{\lambda\rho}^{-1} & -E_{\lambda\rho}^{-1}(BC_{\lambda\rho}^{-1}) \\ -(C_{\lambda\rho}^{-1}B^*)E_{\lambda\rho}^{-1} & C_{\lambda\rho}^{-1}+(C_{\lambda\rho}^{-1}B^*)E_{\lambda\rho}^{-1}(BC_{\lambda\rho}^{-1}) \end{pmatrix}$$

where $E_{\lambda\rho}=A_{\lambda\rho}-BC_{\lambda\rho}^{-1}B^*$ with $A_{\lambda\rho}=A+\lambda\rho$ and $C_{\lambda\rho}=C+\lambda\rho$.

The operator $E_{\lambda\rho}$ becomes a second order (ordinary) elliptic differential operator with coefficients which tend to some positive definite constants as s tends to $-\infty$. Using these concrete expression of the formal resolvent, we can construct the selfadjoint operator $\rho^{-1}K$ in $\mathscr{H}=\{L^2(-\infty,s_0;\rho ds)\}^3$ with resolvent expression (2.6).

§3. Spectral properties of $\rho^{-1}K$

From the resolvent expression (2.6), we can extract the spectral properties of the operator $\rho^{-1}K$ such as the range of the essential spectrum [1],[2],[3],[4] and the absolute continuity of that part [3]. We can also use it in the numerical analysis of the operator [5].

Some of the results are:

1:(flat torus case) the essential spectrum of the Fourier decomposed operator K(k,m) is

$\sigma_{ess}(\rho^{-1}K(k,m))=\sigma_A\cup\sigma_S$, where $\sigma_A = \{\lambda : \lambda = \omega_A(x) \equiv ((n\cos\phi)+(m\sin\phi))^2 b^2/\rho,\ 0 \leqq x < x_0 = \text{the period}\}$

$\sigma_S = \{\lambda : \lambda = \omega_S(x) \equiv \omega(x)_A \gamma P/(b^2+\gamma P),\ 0 \leqq x < x_0\}$.

Furthermore, the essential spectrum contains the absolutely continuous part which is unitarily equivalent to the multiplication operators $\omega_A(x)$ and $\omega_S(x)$.

2:(circular cylinder case) The essential spectrum of the Fourier decomposed operator K(k,m) is also the union of the *ranges of the* multiplication operators $\omega_A(r) \equiv b^2 k_\phi{}^2/\rho$, $0 \leqq r \leqq r_0$, and $\omega_S(r) \equiv \omega_A(r)\gamma P/(b^2+\gamma P)$ $0 \leqq r \leqq r_0$.

We can also investigate more complicated cases such as of general cylindrical or axial symmetricity.

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3次元流における渦度の増大について

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本稿の目的は3次元非粘性流における渦度の増大のメカニズムを渦糸近似を使った現象論的モデルで解明することである。特に、流れの場に有限時間内に渦度の発散する点が現れるかが問題である。

3次元非粘性流に対する渦度方程式は

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} \tag{1}$$

$$\boldsymbol{\omega} = rot\,\mathbf{u} \tag{2}$$

である。(1)の左辺はしばしば

$$\frac{D\boldsymbol{\omega}}{Dt}$$

と書かれ、流れに沿った流体粒子の渦度の時間変化をあらわす。(1)の右辺は速度場の変形テンソルにそった渦度のstretchingまたはcontructionの効果を表している。

2次元流では渦度は実質的にスカラー量で、(1)に対応して

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla u = 0 \qquad \text{または} \qquad \frac{D\omega}{Dt} = 0 \tag{3}$$

すなはち、2次元流では渦のstretchingは起こらず、渦度は流れに沿って流されるだけで、一つの流体粒子の渦度は不変に保たれる。これが2次元流と3次元流の最も大きな違いである。

渦度ベクトル$\boldsymbol{\omega}$の解曲線を渦線、適当な閉曲線にそって渦線を束ねたものを渦管という。特に、有限の断面積の渦管内にだけ渦度がありその外側で渦度が零であるようなものを渦管とよぶことがある。本稿では渦管といえばこのようなものをさすことにする。渦管の断面積が小さく、その半径が渦管の曲率半径に比べて十分小さいものを渦糸という。

一本の渦管の任意の断面Sについての渦度の法線成分の面積分

$$\Gamma = \int_S \boldsymbol{\omega}\cdot\mathbf{n}\, dS \tag{4}$$

を渦管の強さという。これは断面の取りかたによらない。流体が運動するとき渦管も運動するが、Helmholtzの渦定理によれば、渦管は渦管として行動し、かつその強さは不変に保たれる。したがって、流体が運動するにつれて渦管が引き伸ばされれば質量の保存則から渦管の断面積は減少し、渦管のその部分を構成する流体粒子の渦度は増大することになる。

Euler方程式の積分形はいわゆるBiot-Savart則

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi}\int \frac{(\mathbf{x}-\mathbf{x}')\times\boldsymbol{\omega}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d\mathbf{x}' \tag{5}$$

で与えられる。これは、Euler方程式とポテンシャル部分を別にして等価である。(5)を強さΓの渦糸に適用すると

$$\mathbf{u}(\mathbf{x}) = -\frac{\Gamma}{4\pi}\int \frac{(\mathbf{x}-\mathbf{x}')\times ds'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{5'}$$

こゝに、dsは渦糸に沿っての線要素である。

一本の直線渦糸は自分自身の誘起する速度では運動も変形もしない。二本の平行または反平行の直線渦糸の場合には、二本の渦糸は回転運動または平行移動をし、直線は直線のまゝで相対距離も不変である。ところが、一般の曲線渦糸の場合には、渦糸が自分自身の上に誘起する速度は渦糸上の各点で異なるので渦糸は変形

し、変形の結果誘起速度も変わるので、渦糸の変形を追跡する問題は一般に非常に複雑である。

渦糸の変形を調べる際には、しばしば local induction 近似が使われる。これは渦糸上の各点の運動にたいして渦糸の局所的な曲率の効果だけを取り込んだ近似で、その結果によると、渦糸上の点 $\mathbf{x}$ に誘起される速度は

$$\mathbf{u}(\mathbf{x}) = A\,\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}, \qquad A = \frac{\Gamma}{4\pi}\, ln\,\frac{L}{\sigma} \tag{6}$$

の形に与えられる。σ は渦糸の断面の半径である。この近似では渦糸上の各点はその点での陪法線方向に運動するが、渦の stretching は起こらない。したがって、渦の stretching のためには遠距離力の効果を取り入れなければならない。

Siggia (1) は渦糸の衝突という考えから渦度の有限時間内発散の説明を試みている。Siggia によれば渦糸の集合で表せるような流れの場を考えたとき、最近接点ではゞ反平行になっている渦糸の対が現れ、局所的な渦糸対はそれらだけではゞ独立に運動する力学要素とみなされる。曲率を持った反平行の渦対は互いに接近するように運動する。Siggia によれば最近接点が近づくにつれて渦糸の曲率も増大し有限時間内に衝突する。しかし、このモデルでは渦対の最近接点で渦糸が無限大の stretching を起こす説明は不充分である。

本稿では少し違ったメカニズム (2)を考える。渦糸の集合で与えられる流れの場において、一本の渦糸 S に注目する。ある時刻にこの渦糸と局所的に距離が最小の渦糸を L とし、最小の距離を与える S、L 上の点を P_0、Q_0 とする。P_0 と Q_0 の距離を d_0、渦糸の曲率半径、断面の半径をそれぞれ ρ、σ で表す。

$$\rho_0 = min\,(\rho_{P_0}, \rho_{Q_0}), \quad \sigma_0 = max(\sigma_{P_0}, \sigma_{Q_0})$$

としたとき、ある時刻に

$$\rho_0 >> d_0 >> \sigma_0$$

のような状態が実現されたとする。この時、P_0、Q_0付近の流れはρ_0のスケールでは、d_0の距離に置かれた二本の直線渦糸による流れで近似できるであろう。上述のように、平行または反平行の二本の渦糸は変形も相対距離も変えずに平行移動または回転する。しかし、一般の角度で置かれた場合には、一方の渦糸が他方の渦糸上に誘起する速度が場所によって異なるので渦糸は変形し、変形の結果自己誘導速度も現れるので、以後は非常に複雑に変形する。

さて、上の状態から出発して時間τ_0の後に

$$\rho_0 > \rho_1,\ d_0 > d_1,\ \sigma_0 > \sigma_1$$

かつ

$$\rho_1 >> d_1 >> \sigma_1$$

の条件を満たすような点P_1、Q_1が渦糸上のどこかに現れたとしよう。こゝで、幾何学的相似則

$$\rho_1 = \lambda\rho_0,\ d_1 = \lambda d_0,\ \sigma_1 = \lambda\sigma_0 \quad (0 < \lambda < 1)$$

を仮定する。P_1、Q_1付近の流れはρ_1のスケールでは距離がd_1の二本の直線渦糸で近似できる。以後、同様の過程が繰り返されるとすればEuler方程式のスケール則から、有限時間内に渦度の発散する点が現れる。このとき、ある時間t^*を導入すると、渦糸の半径$\sigma(t)$と渦度$\omega(t)$について

$$\sigma(t) \sim (t^* - t)^{1/2}, \qquad \omega(t) \sim (t^* - t)^{-1}$$

が示される。

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準線形性と双曲性の競合効果

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準線形双曲型方程式系の解の構造について一般的な知見を得ようとするのは今日ではまだ無謀なことに違いなく、事実、対称性の高い、具体的な系か、低次元の場合を除き、十分な結果はない。しかし、なめらかな初期値にたいする時間局所解の存在と一意性はよく知られており、さらに、このような解に特異性が発生する最小時間の評価もかなり一般的な観点のもとでなされている。その主要な手法は解の適当なノルムにたいし常微分方程式の比較定理を適用することであり、特異性そのものの性格に立入ることができる場合は依然限られているようである。一方、形式解や漸近解の構成のこころみもなされている。このような近似解候補は性格がわかりやすく、したがって、本来の解との関係が適確に記述できれば、例えば、上述のなめらかな解に発生する特異性の解析には好都合のはずである。ところが、この近似解候補と本来の解との関係が一般にはうまく示されてはいない。

ここでは、線形系との形式的な類似が明白な漸近形式解の構成を紹介し、準線形性の効果の現れるところを見たい。すなわち、準線形の場合でも、「単純波解」を線形の場合の指数関数解に相当するものとして用いれば振動波解に対応する近似解候補が得られることを示す。在来の Choquet-Bruhat, Hunter-Keller などの結果と比べると、展開の第１項がすでにlarge parameter を含む形でありながら小さい項ではないことが大きな違いである。展開の第２項以下がみたすべき方程式の形は、線形系の場合や、準線形系でも上述の Choquet-Bruhat や Hunter-Keller の場合とは違っている。比較のために、この場合の展開も紹介し、その線形化系の場合の展開とも対照して準線形性の効果の現れ方の差異も見るつもりである。剰余項の処理は、近似解候補の展開が十分高い次数まで可能であるとして、なめらかな初期値にたいする Cauchy 問題の解の評価によればよいと考えている。ただし、以上のプログラムを遂行するためには、「単純波解」の属する特性場が純粋非線形であること、初期値と「単純波解」との相性がよいこと、「単純波解」がその導関数をこめて有界にとれること、さらに剰余項のみたすべき方程式系がパラメーターに関して一様な評価を導くものであることなどを要請しなければならない。なお、以上は時間については局所的な議論であるが、空間についての次元の制約は特に要さない（その程度の話に過ぎない）。詳細は講演の際に述べる。

Forced Vibrations for a Superlinear Wave Equation

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We consider the existence of periodic solutions of the following nonlinear vibrating string equation:

$$(1)_{\pm} \qquad u_{tt} - u_{xx} \pm |u|^{p-1}u = f(x,t), \qquad (x,t) \in (0,\pi)\times\mathbb{R},$$

$$(2) \qquad u(0,t) = u(\pi,t) = 0, \qquad t \in \mathbb{R},$$

$$(3) \qquad u(x,t+2\pi) = u(x,t), \qquad (x,t) \in (0,\pi)\times\mathbb{R}.$$

Here, $p > 1$ is a constant and $f(x,t)$ is a 2π-periodic function of t.

Our main result is as follows:

Theorem ([11], c.f. [9,10]). Assume that $p \in (1,\infty)$ and $f(x,t) \in L^{(p+1)/p}_{loc}([0,\pi]\times\mathbb{R})$ is 2π-periodic in t. Then $(1)_{\pm}$-(3) possesses an unbounded sequence of weak solutions in $L^{p+1}_{loc}([0,\pi]\times\mathbb{R})$. That is, for any given $f(x,t)$, there exists a sequence $(u_n)_{n=1}^{\infty} \subset L^{p+1}_{loc}([0,\pi]\times\mathbb{R})$ of weak solutions of $(1)_{\pm}$-(3) such that

$$\int_0^{2\pi}\int_0^{\pi} |u_n(x,t)|^{p+1}dxdt \to \infty \qquad \text{as} \quad n \to \infty.$$

Remarks. 1. (definition of weak solutions). By a weak solution of

$(1)_{\pm}$-(3) we mean a function $u(x,t)$ which is 2π-periodic in t and satisfies

$$\int_0^{2\pi}\int_0^{\pi} [u(\phi_{tt}-\phi_{xx}) \pm |u|^{p-1}u\phi - f\phi]\, dxdt = 0$$

for all smooth ϕ which satisfy (2) and (3).

2. (regularity of weak solutions). If $p = 2n+1$ ($n \in \mathbb{N}$) and $f(x,t)$ is smooth, it is known that any weak solution is also smooth.
3. (existence of free vibrations). In case that $f = 0$ the existence of nontrivial solutions of $(1)_{+}$-(3) has been established by Brezis-Coron-Nirenberg [4] and Rabinowitz [5,7].

To deal with the problem $(1)_{\pm}$-(3) we use variational methods. More precisely we consider the functional $I(u)$:

$$I(u) = \int_0^{2\pi}\int_0^{\pi} [\tfrac{1}{2}(u_t^2 - u_x^2) - \frac{1}{p+1}|u|^{p+1} + fu]\, dxdt \in C^2(E,\mathbb{R}),$$

where E denotes a suitable function space whose elements satisfy (2) and (3). We will seek critical points of $I(u)$. In fact, there is a one-to-one correspondence between critical points of $I(u)$ and weak solutions of $(1)_{+}$-(3).

Recently, Bahri-Lions [2] has studied the problem of the elliptic type:

(4) $\qquad -\Delta u = |u|^{p-1}u + f(x), \qquad x \in D,$

(5) $\qquad u = 0, \qquad x \in \partial D,$

where $D \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary ∂D and $f(x) \in$

$L^{(p+1)/p}(D)$. Dealing with the functional

$$F(u) = \int_D [\tfrac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1} - fu]\, dx \in C^2(H_0^1(D), \mathbb{R}),$$

they have proved the existence of an unbounded sequence of solutions of (4)-(5) under the condition:

$$1 < p < \frac{N}{N-2}.$$

See also Bahri-Berestycki [1], Struwe [8], Rabinowitz [6].

Using the ideas from Bahri-Lions [2] in conjunction with those of Rabinowitz [6,7], we prove our existence theorem for $(1)_\pm$-(3). Here, minimax arguments and energy estimates for $I(u)$ play an important role.

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二次元渦度方程式と渦の形成

Large time behavior of the vorticity of 2D viscous flow and vortex formation in 3D flow

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ABSTRACT

We consider an initial value problem for the two-dimensional vorticity equation and show that the solution $\omega(x,t)$ tends to the Oseen's diffusing vortex at large times keeping the same total vorticity. No particular structure of the initial distribution $\omega(x,0)$ is assumed except the restriction that $R=(1/\nu)\int|\omega(x,0)|\,d^2x$ is small. Applying a time-dependent scale transformation, we show the asymptotic stability of the Burgers' steady vortex. Physically, this implies formation of a concentrated cylindrical vortex.

INTRODUCTION

There exist two well-known exact solutions to the vorticity equation for a viscous incompressible fluid. One is the Oseen's diffusing vortex which is an unsteady axisymmetric solution [1]:

$$\omega_*(x,t) = \frac{\kappa}{4\pi\nu t}\exp[-|x|^2/4\nu t] \qquad (1)$$

where κ is the strength of the vortex, ν the kinematic viscosity and $x(x_1,x_2)$ is the two-dimensional coordinates, the vorticity ω being perpendicular to the plane (x_1, x_2). The other is the Burgers' vortex which is a steady solution [2],

$$\omega_B(x) = \frac{\kappa}{\pi b^2}\exp[-|x|^2/b^2], \qquad b^2=2\nu/a, \qquad (2)$$

in an axisymmetric irrotational flow with a being the parameter characterizing the straining action of the flow. The vorticity in ω_B is concentrated in a radial distance of order $(\nu/a)^{1/2}$ from the axis. The vortex is maintained by the balance of intensification due to the vortex-line stretching by the straining flow and spreading by the viscous diffusion [3].

It is shown here that the Oseen's vortex is asymptotically approached from an arbitrary two-dimensional initial distribution and that a time-dependent scale transformation applied to the asymptotic state leads to the Burgers' vortex. [4]

VISCOUS TWO-DIMENSIONAL FLOW

We consider the initial value problem for the two-dimensional vorticity equation

$$\frac{\partial \omega}{\partial t} - \nu \Delta \omega + (v \cdot \nabla)\omega = 0, \tag{3}$$

$$v = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right) \int \frac{1}{2\pi} \omega(y,t) \log|x-y| \, d^2y \ ,$$

and study the asymptotic behavior of the solution as the time t tends to infinity. Based on the mathematical theory of partial differential equations, it is shown that the vorticity ω is represented asymptotically by ω_* of (1) as $t \to \infty$ even if we start with an arbitrary initial distribution ω_0, provided that

$$R = \frac{1}{\nu} \int |\omega_0| \, d^2x$$

is sufficiently small where $\omega_0 = \omega(x,0)$. The total vorticity is given by

$$K = \int \omega_0(x) \, d^2x \ .$$

More precisely, we prove that for every δ, $0<\delta<1/2$, there is ε such that if $R<\varepsilon$, then the difference of ω from ω_* is estimated as

$$\|\omega - \omega_*\|_p \leqq C \, \| (|x|^2+1)\omega_0\|_1 (\nu t)^{-1+1/p-\delta} \quad \text{for all } t>0 \tag{4}$$

with a universal constant C, where $\|f\|_p$ $(1 \leqq p \leqq \infty)$ denotes L^p-norm in space variables. Since

$$\|\omega_*\|_p = c K (\nu t)^{-1+1/p} \qquad (K \neq 0)$$

with c depending only on p, the estimate (4) gives an asymptotic expression for ω.

This estimate depends on the particular structure of the nonlinear term: the convection term in the two-dimensional vorticity equation (3). We transform the vorticity equation to an integral form and rewrite it for the difference $w = \omega - \omega_*$. Since the convection term vanishes if the vorticity is a function of the radial coordinte $r=|x|$ only, it turns out that w behaves like the solution of the diffusion equation as $t \to \infty$ with the initial data $w(x,0)$, if R is sufficiently small. Since we know the behavior of solution to the diffusion equation, this yields the estimate (4) at least heuristically. To carry out this idea, we need the estimate for

$$\|\omega\|_p \leqq C R t^{-1+1/p} \qquad \text{for } R < 1, \ t > 0 \ , \tag{5}$$

obtained by Giga et al. [5], which itself gives asymptotic behavior of ω in a coarse term. However, the estimate (4) includes an additional fact that if the total net vorticity vanishes, then the vorticity ω decays more rapidly than the estimate (5) owing to the positive term δ in (4).

VORTEX FORMATION

We consider a three-dimensional viscous incompressible flow (space variable y, time τ) represented as a superposition of two flows: an axisymmetric irrotational flow v_1 and a two-dimensional flow v_2 whose vorticity is directed to the symmetry axis. The irrotational part v_1 is assumed to have an radially inward convection and axially stretching flow with a constant rate of strain a: that is, in the cylindrical coordinates (r, ϕ, z) the velocity v_1 is given by

$$v_r = -ar, \qquad v_\phi = 0, \qquad v_z = 2ar \quad .$$

We show that the vorticity field $\Omega(y_*, \tau)$ in this case tends to its equilibrium state of the Burgers' vortex as the time tends to infinity, provided that the Reynolds number of the rotational part (v_2)

$$R = \frac{1}{\nu}\int |\Omega(y_*,0)|\, d^2y_*, \qquad y_*=(y_1,y_2)$$

is sufficiently small where $y=(y_*,y_3)$. This does not restrict the magnitude of the irrotational velocity v_1.

The three-dimensional vorticity equations can be transformed to the two-dimensional problem by the time-dependent scale transformation:

$$x = A(\tau)\, y_*, \qquad t = \frac{1}{2a}(e^{2a\tau} - 1)\ , \qquad A = e^{a\tau}\ , \tag{6}$$

$$\omega(x,t) = A^{-2}(\tau)\, \Omega(y_*,\tau) \quad .$$

The asymptotic behavior of the solution obtained from this transformation is considered by Kambe [6], assuming that the initial vorticity $\Omega(x,0)$ is axially symmetric but with arbitrary R. The governing equation is then reduced to the diffusion equation.

The present results extend this because no particular structure of the initial vorticity is assumed. The asymptotic state represented by $\omega_*(x,t)$ is transformed to

$$\Omega_*(y_*,\tau) = \frac{\kappa}{\pi b^2(1-e^{-2a\tau})}\exp\left[-\frac{|y_*|^2}{b^2(1-e^{-2a\tau})}\right]$$

by the scale transformation (6). Clearly this approaches to $\omega_B(y_*)$ of (2) asymptotically as $\tau \to \infty$.

Physically speaking, our results imply formation of a concentrated vortex from an arbitrary initial field of two-dimensional and rotational velocity under the convective action of the axisymmetric irrotational straining flow, provided R (Reynolds number of the rotational part) is sufficiently small.

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On the Analyticity of Spectral Functions for Some Exterior Boundary Value Porblems

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The present note is concerned with the study of the analytical properties with respect to a spectral parameter k of solutions to the exterior boundary value problems

$$(1)\quad \begin{aligned} (H - k^2)u &= f \quad \text{in} \quad \Omega, \\ Bu &= 0 \quad \text{on} \quad \Gamma, \end{aligned}$$

and the rate of the local energy decay of solutions to the associated dynamic equation

$$(2)\quad \begin{aligned} (\partial_t^2 + H)u(t) &= 0 \quad \text{in} \quad \mathbb{R} \times \Omega, \\ Bu(t) &= 0 \quad \text{on} \quad \mathbb{R} \times \Gamma, \\ u(0) = f_1,\ \partial_t u(0) &= f_2 \quad \text{in} \quad \Omega. \end{aligned}$$

Here Ω is an unbounded domain in $\mathbb{R}^d$ lying in the exterior of its smooth and compact boundary Γ. The H is a symmetric system of second order differential operators defined by

$$Hu(x) = \sum_{m,n=1}^{d} D_m(A_{mn}(x)D_n u(x)), \quad D_m = -i\partial/\partial x_m,$$

where $u \in \mathbb{C}^d$ and $A_{mn}(x)$ are $d{\times}d$ real matrices whose (p,q)-elements $a_{mpnq}(x)$ are C^∞-functions of $x \in \mathbb{R}^d$ and take constant values a^0_{mpnq} outside of a large ball, say for $|x| > b$. The boundary condition we consider is either of Dirichlet or of Neumann type:

$$Bu(x) = u(x) \quad \text{or} \quad Bu(x) = i\sum_{m,n=1}^{d} \nu_m(x)A_{mn}(x)D_n u(x),$$

$\nu(x) = {}^t(\nu_1(x), \ldots, \nu_d(x))$ being the unit outward normal to Γ at $x \in \Gamma$. We assume that the dimension d satisfies

(A.1) $\quad d \geq 3.$

We demand the following conditions on the coefficients $A_{mn}(x)$.

(A.2)(Hyperelasticity)

$$a_{mpnq}(x) = a_{pmnq}(x) = a_{nqmp}(x), \quad x \in \mathbb{R}^d.$$

(A.3)(Stability) There exists a constant $C > 0$ such that the inequality

$$\sum_{m,p,n,q=1}^{d} a_{mpnq}(x) s_{nq} \bar{s}_{mp} \geq C \sum_{m,p=1}^{d} |s_{mp}|^2$$

is valid for any $x \in \mathbb{R}^d$ and $d \times d$ Hermitian matrix $s = (s_{mp})$.

(A.4) Let $A^0_{mn} = (a^0_{mpnq})$ and $A(\xi) = \sum_{m,n=1}^{d} A^0_{mn} \xi_m \xi_n$. The characteristic roots of $A(\xi)$ are of constant multiplicity for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

In order to state the results we introduce some notation and functional spaces. We set

$$\begin{aligned} D &= \mathbb{C} && \text{when } d \text{ is odd,} \\ &= \{k \in \mathbb{C}\setminus\{0\};\ -\tfrac{3}{2}\pi < \arg k < \tfrac{\pi}{2}\} && \text{when } d \text{ is even;} \end{aligned}$$

$$L^2_a(G) = \{u \in L^2(G);\ u(x) = 0 \ \text{ for } \ |x| \geq a\},\ a > 0;$$

$$H^2_e(G) = \{u \in H^2_{loc}(G);\ \exp(-|x|^2)D^\alpha u \in L^2(G),\ |\alpha| \leq 2\};$$

$$\begin{aligned} \hat{H}^p(G) = \{u \in H^p_{loc}(G);\ & D^\alpha u \in L^2(G),\ 1 \leq |\alpha| \leq p, \\ & \varliminf_{R\to\infty} R^{-2}\int_{R<|x|<2R} |u(x)|^2\, dx = 0\}. \end{aligned}$$

Let $a > 0$ be a fixed number so that $b < a$, and $\mathbb{B}$ be the totality of bounded linear operators of $L^2_a(\Omega)$ into $H^2_e(\Omega)$.

Theorem 1 ([2], [3]). *Suppose that* (A.1)-(A.4) *are valid. Then there exists an operator* $R(k) \in \mathbb{B}$ *such that* $R(k)$ *depends meromorphically on the parameter* $k \in D$ *and satisfies the following properties: Let* Λ *stand for the set of poles of* $R(k)$ *in* D.

(i) Λ *is discrete.*

(ii) *If* $k \in D \setminus \Lambda$ *and* $f \in L^2_a(\Omega)$, *then* $u = R(k)f$ *solves the boundary value problem* (1).

(iii) $\Lambda \cap \{k \in D;\ \mathrm{Im}\, k < 0\} = \phi$.

(iv) *If* $k \in D$, $\mathrm{Im}\, k < 0$, *then* $R(k)f \in H^2(\Omega)$ *for* $f \in L^2_a(\Omega)$.

(v) *If* $k \in \Lambda \cap (\mathbb{R} \setminus \{0\})$, *then there exists a non-trivial function* $u(x) \in C_0^\infty(\bar{\Omega}_b)$ *satisfying* (1) *with* $f = 0$, *where* $\Omega_b = \{x \in \Omega; |x| \leq b\}$.

(vi) *Put* $U_r = \{k \in D;\ |k| < r\}$. *There exists a constant* $K > 0$ *such that*

1° *if* d *is odd,* $R(k)$ *is a* $\mathbb{B}$*-valued holomorphic function of* $k \in U_K$;

2° *if* d *is even,* $R(k)$ *is described as*

$$R(k) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} G_{mn}(k^{d-2} \log k)^m k^n, \quad k \in U_K,$$

where $G_{mn} \in \mathbb{B}$ *and the double series converges in the operator norm uniformly for* $k \in U_K$.

In case $A_{mn}(x)$ equal identically to A^0_{mn}, we can use the unique continuation theorem to have

Corollary 2. *Let* $A_{mn}(x) \equiv A^0_{mn}$. *Then* $\Lambda \cap (\mathbb{R}\setminus\{0\}) = \phi$.

Our strategy in proving Theorem 1 follows that of Vainberg [7], where the analytical properties in the spectral parameter is discussed for solutions to certain class of elliptic problems in the whole space or in an exterior domain (see also [8]). Tsutsumi [6] also proves the results corresponding to Theorem 1, (vi) for the operator $-\Delta - k^2$ with Dirichlet boundary condition. The maximum principle plays an essential role in his method. When $d = 3$, H is the elastic equation, and $C\Omega$ is convex, then Dassios and Kiriaki study in [1] the asymptotic behaviour of solutions with respect to small k by essential use of the representation of the Green function. Our idea of proving Theorem 1, (vi) is, however, completely different from those of [1], [6], [8], and close to that due to Shibata [4, Part I].

Sketch of Proof of Theorem 1. Let $H_0 = \sum_{m,n=1}^{d} A^0_{mn} D_m D_n$. Then the Green function of $H_0 - k^2$, $\mathrm{Im}\, k < 0$ has an analytic extension $R_0(k)$ in the complex plane of parameter k which belongs to $\mathbb{B}(L^2_a(\mathbb{R}^d), H^2_e(\mathbb{R}^d))$. Let L_0 be the Green operator to the interior boundary value problems

$$Hu = f \quad \text{in} \quad \Omega_a,$$
$$Bu = 0 \quad \text{on} \quad \Gamma, \; u = 0 \quad \text{on} \quad |x| = a.$$

Then $L_0 \in \mathbb{B}(L^2(\Omega_a), H^2(\Omega_a))$. We define a regularizer $G(k)$ as follows:

$$G(k)f = \varphi R_0(k)f_0 + (1-\varphi)L_0 f, \quad f \in L^2_a(\Omega),$$

where $\varphi \in C^\infty(\mathbb{R}^d)$, $0 \le \varphi \le 1$, $\varphi = 1$ for $|x| \ge R_2$, and $= 0$ for $|x| < R_1$ with $b < R_1 < R_2 < a$; $f_0 = f$ in Ω, and $= 0$ in $\complement\Omega$. Then $G(k)f$ satisfies the boundary condition on Γ. If we denote by $S(k)$ the operator defined by $S(k)f = (H - k^2)f - f$, then $S(k)$ is a compact operator in $L^2_a(\Omega)$ with an analytic parameter $k \in D$. We prove the key result that the bounded inverse $(I + S(0))^{-1}$ exists. If we take account of the continuity of $S(k)$ in $k \in D \cup \{0\}$, we can apply the analytic Fredholm theorem to obtain the existence of the meromorphic bounded inverse $(I + S(k))^{-1}$. The operator $R(k) = G(k)(I + S(k))^{-1}$ turns out to be the very one we desired. □

From now on we always assume that $A_{mn} = A^0_{mn}$. We shall apply Theorem 1 and Corollary 2 to discuss the rate of local energy decay of solutions to (2). We set $f = {}^t(f_1, f_2)$ and

$$\mathcal{H} = \{ f;\ f_1 \in \dot{H}^1(\Omega),\ f_2 \in L^2(\Omega).\ \ f_1 = 0 \ \text{ on } \Gamma \ \text{ if } \ B = 1\}.$$

The energy norm in $\mathcal{H}$ is defined as follows:

$$\|f\|^2_{E,\Omega} = \sum_{m,n=1}^{d} \int_\Omega A^0_{mn} D_n f_1 \cdot \overline{D_m f_1}\, dx + \int_\Omega |f_2|^2 dx.$$

Define by $Lf = {}^t(f_2, -H_0 f_1)$ the operator L in $\mathcal{H}$ with domain

$$\mathcal{D}(L) = \{f \in \mathcal{H};\ f_1 \in \dot{H}^2(\Omega),\ f_2 \in \dot{H}^1(\Omega) \cap L^2(\Omega),$$
$$Bf_1 = 0 \text{ on } \Gamma.\ f_2 = 0 \text{ on } \Gamma \text{ if } B = 1\}.$$

Then Shibata and Soga show in [5] that $\mathcal{H}$ is a Hilbert space equiped with norm $\|\cdot\|_{E,\Omega}$ and that L is a skew selfadjoint operator in $\mathcal{H}$. Stone's theorem guarantees that L generates a one-parameter unitary group $U(t)$ in $\mathcal{H}$. We write $U(t)f = {}^t(u(t),\ \partial_t u(t))$, $f \in \mathcal{H}$. Then $u(t)$ is a unique solution to (2) with initial data $f_1 \in \dot{H}^1(\Omega)$ and $f_2 \in L^2(\Omega)$.

Definition 3. We say that Ω is non-trapping if there exists $T > 0$ such that $U(t)f$ belongs to $C^\infty([0, \infty) \times \overline{\Omega}_a)$ for any $f = {}^t(0, f_2)$, $f_2 \in L^2_a(\Omega)$.

We use $R(k)$ to describe $u(t)$ by the inverse Laplace transform and properly shift the contour. Then we are led to the following result through a routine calculus due to Vainberg [9].

Theorem 4. *Assume that* Ω *is non-trapping. Let* $f \in \mathcal{H}$ *and vanish for* $|x| > a$. *Then we have the local energy estimate*

$$\|U(t)f\|_{E,\Omega_a} \le p(t)\|f\|_{E,\Omega},$$

where $p(t)$ *is given by*

$$
\begin{aligned}
p(t) &= C_1\exp(-C_2|t|) \quad &\text{if } d \text{ is odd,}\\
&= C_1(1+|t|)^{-(d-1)} \quad &\text{if } d \text{ is even,}
\end{aligned}
$$

with positive constants C_1 *and* C_2 *being independent of* t, f.

We conclude this note by discussing the elastic wave equation as a typical example. Let $d = 3$ and an exterior domain Ω be filled with an isotropic and homogeneous elastic medium specified with the Lamé constants λ and μ. Consider the elastic equation with a parameter k:

$$(P_0 - k^2)u = f; \; P_0u = -\mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u),$$

where we assume that $\mu > 0$ and $3\lambda+2\mu > 0$. One can easily see that P_0 satisfies the assumptions (A.1)-(A.4). Further, let $\complement\Omega$ is convex and $B = 1$. Then Yamamoto [10] proves that Ω is non-trapping, which together with Theorem 4 allows us to have the exponential local energy decay for solutions to the elastic wave equation $(\partial_t^2 + P_0)u(t) = 0$ with Dirichlet boundary condition.

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WELL POSEDNESS FOR QUASI-LINEAR HYPERBOLIC MIXED PROBLEMS WITH CHARACTERISTIC BOUNDARY

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§ 1. Introduction and results.

Let G be a domain in R^n with smooth and compact boundary ∂G. We consider the following mixed problem for symmetrizable hyperbolic system P under the conditions (I)～(V) described below :

$$(P,B)\qquad \begin{cases} Pu \equiv (D_t + \sum_{j=1}^{n} A_j D_j + C)u = f & \text{in } [t_1, t_2] \times G, \\ Bu = g & \text{on } [t_1, t_2] \times \partial G, \\ u(t_1, x) = h & \text{for } x \in G, \end{cases}$$

where $D_t = -i\frac{\partial}{\partial t}$, $D_j = -i\frac{\partial}{\partial x_j}$.

(I) A_j and C are $m \times m$ matrices and $A_0 A_j = (A_0 A_j)^*$ with a hermitian matrix $A_0 \geq a_0$ (positive constant), furthermore

$$A_j, C \in X_q([t_1, t_2]; G)$$

$$= \bigcap_{i=0}^{q} C^i([t_1, t_2]; H^{q-i}(G)) \equiv X_q,$$

where $q = \max(p, [\frac{n}{2}] + 2)$ with $p \geq 0$ and $C^i([t_1, t_2]; H^k(G))$ is of class C^i on $[t_1, t_2]$ to $H^k(G)$, k-th order Sobolev space in G.

(II) $B(t,x)$ is a $d^{+}\times m$ C^{∞}-matrix of constant rank d^{+}.

(III) (i) ∂G is characteristic for P, that is, the boundary matrix A_ν is of constant rank d less than m on ∂G, where for x near ∂G $A_\nu = \sum_{j=1}^{n} A_j \nu_j$ and $\nu(x) = (\nu_1, \cdots, \nu_n)$ stands for the unit inward normal to ∂G at the boundary point nearest to $x \in \overline{G}$.

(ii) The eigenvalues of A_ν are of constant multiplicity near ∂G.

(IV) The kernel B is maximally nonpositive for $A_0 A_\nu$ on ∂G, i.e.,

$$A_0 A_\nu u \cdot u \leq 0 \quad \text{for } u \in \ker B \quad \text{on } \partial G \tag{1.1}$$

and ker B is a maximal subspace obaying this property. Note that this implies the number of positive eigenvalues of A_ν is d^{+} on ∂G.

Now, since A_ν is of rank d on ∂G and $A_0 A_j \in X_q$ are hermitian, there exist (see Lemma 7.4 below) positive numbers a_1, a_2 and an $m\times m$ nonsingular matrix W defined in a neighborhood of each boundary point which belongs to X_q and satisfies the followings there:

$$W^{-1} A_\nu W = \begin{pmatrix} A_{\mathrm{I}} & 0 \\ 0 & A_{\mathrm{II}} \end{pmatrix}, \ \det A_{\mathrm{I}} \neq 0 \quad \text{and } A_{\mathrm{II}}|_{\partial G} = 0, \tag{1.2}$$

$$|A_{\mathrm{I}}^{-1}| \leq a_1, \quad |W^{-1}| \leq a_2, \tag{1.3}$$

where A_{I}, A_{II} are $d\times d$, $(m-d)\times(m-d)$ real diagonal matrices respectively:

$$A_{\mathrm{I}}(t,x) = \begin{pmatrix} a_{\mathrm{I}}^{1} & & 0 \\ & \ddots & \\ 0 & & a_{\mathrm{I}}^{d} \end{pmatrix}, \quad A_{\mathrm{II}}(t,x) = \begin{pmatrix} a_{\mathrm{II}}^{1} & & 0 \\ & \ddots & \\ 0 & & a_{\mathrm{II}}^{m-d} \end{pmatrix}. \tag{1.4}$$

For x near ∂G, take a vector $\eta(x) = (\eta_1, \cdots, \eta_n) \in \{\nu(x)\}^{\perp}$, the orthogonal subspace to $\nu(x)$ and write

$$(1.5) \qquad \sum_{j=1}^{n}(W^{-1}A_jW)(t,x)\eta_j(x) = \begin{pmatrix} A_{II} & A_{I\mathrm{II}} \\ A_{\mathrm{II}I} & A_{\mathrm{II}\mathrm{II}} \end{pmatrix}(t,x;\eta(x)),$$

$$= \sum_{j=1}^{n} A_{JK}^{j}(t,x)\eta_j(x),$$

where A_{II}, $A_{\mathrm{II}\mathrm{II}}$ are $d\times d$, $(m-d)\times(m-d)$ matrices respectively. Now let $\mathcal{A}(t,x;\eta(x))$ be an $(m-d)\times d$ matrix defined uniquely (see Lemma 7.1 below) by the equation

$$(1.6) \qquad \mathcal{A}A_I - A_{\mathrm{II}}\mathcal{A} = A_{\mathrm{II}I},$$

that is (if (1.4) holds),

$$(1.7) \qquad \mathcal{A} = (b_{ik}(a_I^k - a_{\mathrm{II}}^i)^{-1}) \qquad \text{where} \quad (b_{ik}) = A_{\mathrm{II}I}.$$

And we pose the following condition (V) on $\{A_j\}$ (and G):

(V) For x near ∂G

$$(\mathcal{A})_G : (\mathcal{A}A_{II}-A_{\mathrm{II}\mathrm{II}}\mathcal{A})(t,x;\eta(x)) = (\mathcal{A}A_{I\mathrm{II}})(t,x;\eta(x)) = 0 \quad \text{for } \eta(x)\in\{\nu(x)\}^{\perp}.$$

Remark that (V) is a pointwise condition, and that does not depend on the choice of W even if we take W as in (1.2) for which A_I, A_{II} are not diagonal, define $A_{JK}(J,K=\mathrm{I},\mathrm{II})$, $\mathcal{A}$ and (V) by the above procedure, because another W can be obtained by multiplying W by a nonsingular matrix with the same blocked form as $\begin{pmatrix} A_I & 0 \\ 0 & A_{\mathrm{II}} \end{pmatrix}$.

We shall say that the data f, g and h satisfy compatibility conditions for (P,B) of order k at $t=t_1$, if

$$\sum_{i=0}^{j}\binom{j}{i}(D_t^{j-i}B)(t_1)\text{“}D_t^iu(t_1)\text{”} = D_t^ig(t_1) \quad \text{on } \partial G \text{ for } 0\le i\le k,$$

where "$D_t^i u(t_1)$" is the function defined by only f and h which is obtained by solving the equations for $D_t^i u(t_1): u(t_1)=h$ and $Pu=f$ near t_1. Then under our assumptions (I)～(V) on the operators $\{P,B\}$ we have

<u>Theorem 1.</u> Let $p \geq 0$. Then there exist a with $0 \leq a < 1$ and M_{q-1}, M_q, $\gamma_q > 0$ such that for every $f \in H_p([t_1, t_2] \times G)$, $g \in \bigcap_{i=0}^{p+1} C^i([t_1, t_2]; H^{p+1/2-i}(\partial G))$ and $h \in H^p(G)$ satisfying compatibility conditions of order p-1 at $t=t_1$, there exists a unique solution $u \in X_p([t_1, t_2]; G)$ to (P,B) such that for any $\gamma \geq \gamma_q$ and $t \in [t_1, t_2]$

$$
\begin{aligned}
(1.8) \qquad & e^{-\gamma t} |||u(t)|||_p^2 + \gamma \int_{t_1}^{t} e^{-\gamma t} |||u(s)|||_p^2 ds \\
& \leq (1 + |\{A_j\}|_{X_q}^{2a}) M_{q-1} e^{-\gamma t_1} (|h|_p^2 + |||f(t_1)|||_{p-1}^2 \\
& \qquad + \sum_{i=0}^{p} \langle D_t^i g(t_1) \rangle_{p-1/2-i}^2) \\
& \quad + M_q \gamma^{-1} \int_{t_1}^{t} e^{-\gamma s} (|||f(s)|||_p^2 + \sum_{i=0}^{p+1} \langle D_s^i g(s) \rangle_{p+1/2-i}^2) ds,
\end{aligned}
$$

where a can be taken zero if $p \neq [\frac{n}{2}]+2$ and γ_k, M_k are positive, continuons and nondecreasing functions of $(|\{A_j\}, C|_{X_k}, a_0^{-1}, a_1, a_2)$ which depend on B, t_1, t_2 and G but not on f, g, h. Here the norms are defined as

$$
(1.9) \qquad |u|_{X_k} = \sup_{t_1 \leq t \leq t_2} |||u(t)|||_k,
$$

$$
(1.10) \qquad |||u(t)|||_k^2 = \sum_{i=0}^{k} |D_t^i u(t)|_{k-i}^2 \equiv |||u(t)|||_{k,G}^2
$$

and $|\cdot|_j=|\cdot|_{j,G}$, $\langle\cdot\rangle_j=\langle\cdot\rangle_{j,\partial G}$ stand for the norms of $H^j(G)$, $H^j(\partial G)$ respectively.

We next consider the case where Pu is quasi-linear in u under the conditions $(0)'\sim(V)'$ described below :

$$
(P(u),B)\quad \begin{cases} P(u)u \equiv (D_t+\sum_{j=1}^{n}A_j(t,x,u)D_j+C(t,x,u))u=f & \text{in } [0,T]\times G, \\ B(x)u = 0 & \text{on } [0,T]\times\partial G, \\ u(0,x) = h & \text{for } x\in G. \end{cases}
$$

$(0)'$ $p\geq[\frac{n}{2}]+2$, $h\in H^p(G)$ is real and $Bh=0$ on ∂G.

Let ε_0, δ_0, $T_0>0$ and set

$$
(1.11)\qquad N_{\varepsilon_0} = \{(x,v)\in\bar{G}\times R^m; |v-h(x)| \leq \varepsilon_0\},
$$

$$
N_{\varepsilon_0}^{\delta_0} = \{(x,v)\in U_{\delta_0}(\partial G)\times R^m; |v-h(x)| \leq \varepsilon_0\},
$$

where $U_{\delta_0}(\partial G)$ is a δ_0-neighborhood of ∂G. Then $N_{\varepsilon_0}^{\delta_0}\subset N_{\varepsilon_0}$ and $N_{\varepsilon_0}^{\delta_0}$ is a neighborhood in $G\times R^m$ of the set $\{(x,h(x))\}_{x\in\partial G}$. We now pose the following conditions $(I)'\sim(VI)'$ on the problem (P(u),B) which are satisfied by some fluid dynamics examples as in section 2:

$(I)'$ (i) $A_j(t,x,v)$ and $C(t,x,v)$ are $m\times m$ matrices defined in $[0,T_0]\times N_{\varepsilon_0}$ which have bounded continuous derivatives of up to p-th order.

(ii) There exist $a_0>0$ and $A_0(t,x,v)$ satisfying (i) with j=0 such that A_0 and A_0A_j are symmetric and $A_0\geq a_0$ for all $(t,x,v)\in[0,T_0]\times N_{\varepsilon_0}$

$(II)'$ $B(x)$ is a $d^+\times m$ C^∞-matrix of constant rank d^+.

$(III)'$ (i) rank $A_\nu(t,x,v)=d=\text{const.}<m$ for all (t,x,v) $\in[0,T_0]\times N_{\varepsilon_0}^{\delta_0}$ such that $x\in\partial G$ and $B(x)v=0$.

(ii) The eigenvalues of $A_\nu(t,x,v)$ are of constant multiplicity for all $(t,x,v)\in[0,T_0]\times N_{\varepsilon_0}^{\delta_0}$.

(IV)′ ker $B(x)$ is maximally nonpositive for $(A_0A_\nu)(t,x,v)$ for all (t,x,v) as in (III)′(i).

(V)′ $\{A_j(t,x,v)\}$ satisfies $(\mathcal{A})_G$ for all (t,x,v) $\in[0,T_0]\times N_{\varepsilon_0}^{\delta_0}$.

Under our assumptions (0)′∼(V)′ on the problem $(P(u),B)$ we have

<u>Theorem 2.</u> Let $h\in H^p(G)$ and $f\in H^p([0,T]\times G)$ be real and satisfied compatibility conditions for $(P(u),B)$ of order $p-1$ at $t=0$. Then $(P(u),B)$ has a unique solution $u\in X_p([0,T];G)$ if T is sufficiently small.

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